1 Local Heights: Wrap up

Last time we defined local heights by

$$\lambda_{\mathcal{D}}(P) = \max_k \min_l \log \left| \frac{s_k}{t_l s_D}(P) \right|$$

where $\mathcal{D} = (s_D, \mathscr{L}, \mathbf{s}, \mathscr{M}, \mathbf{t})$ is a presentation of Cartier divisor D.

Our goal is to study how the presentation affects the height, and it turns out it is bounded by a constant.

As usual we will fix an absolute value $|\cdot|$ on $\overline{K}$.

Definition 1.1

Let $U \subseteq \mathbb{A}^n$ be closed. A set $E \subseteq U(\overline{K})$ is ***bounded*** in U if for any

$$f \in K[U] = K[x_1, ..., x_n]/\sqrt{\mathscr{I}(U)}$$

, the function $|f|$ is bounded on E, i.e. $\{|f(u)| : u \in E\}$ is a bounded set in $\mathbb{R}$.

Lemma 1.2

Let $\{f_1, ..., f_N\}$ be generators of $K[U]$. If

$$\sup_{P \in E} \max_{j=1,...,N} |f_j(P)| < \infty$$

then E is bounded.

Proof. Take $f = p(f_1, ..., f_N)$ where p is a polynomial in $K[U]$. Let C be the number of monomials in p and d the degree of p. Define $\delta = 1$ if the place is archimedean and 0 otherwise. Then let $|p| := \max_j |a_j|$ where a_j range over all coefficients of p. Then we see

$$\sup_{P \in E} |f(P)| \leq C^{\delta} |p| \cdot \max(1, \sup_{P \in E} \max_{j=1,...,N} |f_j(P)|)^d < \infty$$

Q.E.D.

Lemma 1.3

If $\{U_i\}$ is finite affine open cover of affine K-variety U and E is bounded in U. Then there are bounded subsets E_i of U_i such that $E = \bigcup E_i$.

Proof. WLOG we may assume $U_i = \{x \in U : h_i \neq 0\}$ for some $h_i \in K[U]$. Define $E_i := \{P \in E : |h_i(P)| = \max_k |h_k(P)|\}$. Say $f_1, ..., f_N$ generates $K[U]$, then $f_1, ..., f_N, 1/h_i$

generates U_i, and thus it suffices to prove $1/h_i$ is bounded on E_i. Now use the fact there exists regular functions g_i so $\sum g_i h_i = 1$ to show

$$\sup_{P \in E_i} |1/h_i(P)| \leq C^\delta \sup_{P \in E} \max_k |g_k(P)| < \infty$$

Theorem 1.4

Let X be projective variety over K, $\mathcal{D}, \mathcal{D}'$ be two presentations of Cartier divisor D. Then there exists γ so

$$|\lambda_{\mathcal{D}}(P) - \lambda_{\mathcal{D}'}(P)| < \gamma$$

for all $P \notin \operatorname{supp}(D)$.

Proof. Note λ is functorial, thus $\lambda_{\mathcal{D}} - \lambda_{\mathcal{D}'} = \lambda_{\mathcal{D}-\mathcal{D}'}$. That is, $\lambda_{\mathcal{D}} - \lambda_{\mathcal{D}'}$ is a local height with respect to $D - D$, i.e. it suffices to prove the claim for D being trivial and one of the presentation, say $\mathcal{D}'$, is equal $(1, \mathscr{L}, 1, \mathscr{M}, 1)$. Thus $\mathcal{D} = (1, \mathscr{L}, \mathbf{s}, \mathscr{M}, \mathbf{t})$. We need to find γ so

$$-\gamma \leq \max_k \min_l \log |s_k/t_l(P)| \leq \gamma$$

To that end, it suffices to prove

$$\max_k \min_l \log |s_k/t_l(P)| \leq \gamma$$

as we can exchange the role of $\mathbf{s}$ and $\mathbf{t}$.

Choose embedding $X \hookrightarrow \mathbb{P}^N$ with coordinates $(x_0 : \ldots : x_N)$, and $U_i = \{x \in X : x_i \neq 0\}$. Now define $U_{il} = \{x \in U_i : t_l(x) \neq 0\}$. Then $g_{kl} := s_k/t_l$ restricts to U_{il} are regular functions. The functions $f_{ij} = x_j/x_i$ generates $K[U_i]$. Now define

$$E_i = \{P \in X(\overline{K}) : |x_i(P)| = \max_j |x_j(P)|\}$$

and its clear for $P \in E_i$, we have $\max_j |f_{ij}(P)| = 1$, hence E_i is bounded in U_i by lemma above. Thus we can apply another lemma to U_i and E_i to obtain bounded subset $E_{il} \subseteq U_{il}$ and

$$\sup_{P \in E_{il}} \max_k |g_{kl}(P)| < \infty$$

Since E_{il} covers $X(\overline{K})$, we are done.

2 Global Heights

Now let us fix number field K, then for each place v and Cartier divisor D with presentation $\mathcal{D}$, we define

$$\lambda_{\mathcal{D}}(P, v) = \max_k \min_l \log \left|\frac{s_k}{t_l s_D}(P)\right|_v$$

Now let $p \in M_{\mathbb{Q}}$ be the place lying below v, and $u \in M_{\overline{K}}$ extend v, then

$$\lambda_{\mathcal{D}}(P, v) = \frac{[F_v : \mathbb{Q}_p]}{[F : \mathbb{Q}]} \lambda_{\mathcal{D}}(P)$$

where the $\lambda_{\mathcal{D}}(P)$ is the local height with respect to u defined in our last talk. Thus, we can apply all the results we developed for local heights to $\lambda_{\mathcal{D}}(P, v)$.

Example 2.1

Consider $D = \{x_0 = 0\}$ in $\mathbb{P}^n$ with presentation

$$\mathcal{D} = (x_0, \mathscr{O}(1), \{x_0, ..., x_n\}, \mathscr{O}, 1)$$

For $P \in \mathbb{P}^n(F)$ with $x_0(P) \neq 0$ and $v \in M_F$, we have

$$\lambda_{\mathcal{D}}(P, v) = \max_k \log |\frac{x_k}{x_0}(P)|_v$$

and thus product formula implies

$$h(P) = \sum_{v \in M_F} \lambda_{\mathcal{D}}(P, v)$$

where $h(P)$ is the height on projective space.

Now let $\lambda_{\mathcal{D}}$ be a local height relative to the presentation $\mathcal{D} = (s_D, \mathscr{L}, \mathbf{s}, \mathscr{M}, \mathbf{t})$ of Cartier divisor $D = (U_i, f_i)$. For $P \in X$, we want to define the global height with respect to D, which should be intuitively just sum $\lambda_{\mathcal{D}}(P, v)$ over all the places in M_K. Thus, we want to twist D so that P always not in the support of the divisor.

Specifically, for $P \in X$, since $(\mathscr{L}, \mathbf{s})$ an $(\mathscr{M}, \mathbf{t})$ both gives embedding of X into some projective space, there are s_j and t_l so $s_i(P) \neq 0$ and $t_l(P) \neq 0$. Thus, we can find non-zero meromorphic section $s : X \to \mathscr{O}(D)$ of $\mathscr{O}(D)$ so P is not contained in the support of the Cartier divisor $D(s) := (U_i, f_i \circ s)$ with presentation $\mathcal{D}(s) = (s, \mathscr{L}, \mathbf{s}, \mathscr{M}, \mathbf{t})$, and we have

$$\lambda_{\mathcal{D}(s)} = \lambda_{\mathcal{D}} + \lambda_{s/s_D}$$

Definition 2.2

In the situation above, the ***global height*** of $P \in X(F)$ relative to $\lambda = \lambda_{\mathcal{D}}$ is defined by

$$h_\lambda(P) = \sum_{v \in M_F} \lambda_{\mathcal{D}(s)}(P, v)$$

Proposition 2.3

The global height h_λ is independent of F and the section s.

Proof. It is clear this definition is independent of F. Now let t be another non-zero meromorphic section of $\mathcal{O}(D)$ with $P \notin \mathrm{supp}(D(t))$. Then we see

$$\lambda_{\mathcal{D}(s)}(P,v) - \lambda_{\mathcal{D}(t)}(P,v) = \lambda_{s/t}(P,v)$$

for any $v \in M_F$. But the product formula says

$$\sum_{v \in M_F} \lambda_{s/t}(P,v) = 0$$

and so we are done.

Theorem 2.4

Let λ, λ' be local heights relative the Cartier divisors D, D', with $D - D'$ a principal divisor. Then $h_\lambda - h_{\lambda'}$ is a bounded function.

This roughly follows from the fact $\lambda_{\mathcal{D}} - \lambda_{\mathcal{D}'}$ is bounded by a constant.

Theorem 2.5

The map $h : \mathrm{Pic}(X) \to \mathbb{R}^X / \mathcal{O}(1)$ from Picard group of X to the space of functions from X to $\mathbb{R}$ quotient by bounded functions is a homomorphism of groups. If $\phi : Y \to X$ is a morphism of irreducible projective varieties over K, then

$$h_{\phi^* \mathscr{L}} = h_{\mathscr{L}} \circ \phi$$

3 Weil Heights

Let X be projective over $\overline{\mathbb{Q}}$.

Definition 3.1

Let $\phi : X \to \mathbb{P}^n_{\overline{\mathbb{Q}}}$ be a morphism over $\overline{\mathbb{Q}}$. The ***Weil height*** of $P \in X(\overline{\mathbb{Q}})$ relative to ϕ is defined by $h_\phi(P) := h \circ \phi(P)$, where h is the usual height on $\mathbb{P}^n_{\overline{\mathbb{Q}}}$.

Now let $\phi : X \to \mathbb{P}^n$ and $\psi : X \to \mathbb{P}^m$, then we define the ***join*** $\phi \# \psi$ as the morphism

$$s_{n,m} \circ (\phi \times \mathrm{Id}) \circ \Gamma(\psi)$$

where $s_{n,m}$ is the Segre embedding, $\phi \times \mathrm{Id}$ a fiber product, $\Gamma(\psi)$ the graph of ψ. Explicitly,

$$\phi \# \psi : X \to \mathbb{P}^{(n+1)(m+1)-1}, \quad x \mapsto (\phi_i(x)\psi_j(x))$$

Remark 3.2

If ϕ is a closed embedding, then $\phi\#\psi$ is closed embedding. Indeed, $\Gamma(\psi)$ is always closed embedding as X is assumed to be separated (separated=closed diagonal=closed graph=closed equalizer). The Segre embedding is always closed, and closed immersion is stable under base change and composition, hence we are done.

Proposition 3.3

If $\phi : X \to \mathbb{P}^n$ and $\psi : X \to \mathbb{P}^m$ are morphisms, then

$$h_{\phi\#\psi} = h_\phi + h_\psi$$

Now we will show global heights=difference of Weil heights.

Say we have $\phi : X \to \mathbb{P}^n$, then there is a linear form $\ell = \sum \ell_i x_i$ which does not vanish identically on irreducible components of X. Then we see h_ϕ is the global height associated with $\phi^*(\ell, \mathscr{O}_{\mathbb{P}^n}, \{x_0, ..., x_n\}, \mathscr{O}_{\mathbb{P}^n}, 1)$. Conversely, say we have $\mathcal{D} = (s, \mathscr{L}, \mathbf{s}, \mathscr{M}, \mathbf{t})$, then $(\mathscr{L}, \mathbf{s})$ and $(\mathscr{M}, \mathbf{t})$ both induces closed immersions ϕ, ψ of X into projective spaces, and its not hard to see $h_{\mathcal{D}} = h_\phi - h_\psi$.

Theorem 3.4: Northcott

Let X be projective variety defined over a number field K, $h_{\mathscr{L}}$ a height function associated with ample $\mathscr{L} \in \mathrm{Pic}(X)$. Then the set

$$\{P \in X(\overline{K}) : h_{\mathscr{L}}(P) \leq C, [\kappa(P) : K] \leq d\}$$

is finite for any constant $C, d \in \mathbb{R}$.

Proof. This follows from Northcott theorem for heights on projective space.

The last thing we will do is explicit bound on the difference of two Weil heights.

Let X be irreducible projective over $\overline{\mathbb{Q}}$ of dimension r. Then there is a $\overline{\mathbb{Q}}$-morphism $\pi : X \to \mathbb{P}^{r+1}$ such that X is mapped birationally onto a hypersurface.

Explanation

Note $K(X)$ is separable over K, thus $K(X)$ is finite dimensional $K(f_1, ..., f_r)$-vector space, where f_i are algebraically independent. Thus $K(X)$ is generated by a rational function f_{r+1} over $K(f_1, ..., f_r)$. Now let p be the minimal polynomial of f_{r+1} over $K(f_1, ..., f_r)$, and assume $p = q(f_1, ..., f_r)$ with $q \in K[x_1, ..., x_{r+1}]$. Then $V(q)$

is birational to X.

Now let $z_0, \ldots, z_{r+1}$ be the standard coordinates of $\mathbb{P}^{r+1}$, then we may assume the hypersurface is given by $V(f)$ where f is irreducible homogeneous of degree d, and it is given by

$$f(z_0, \ldots, z_{r+1}) = f_0 + f_1 z_{r+1} + \ldots + f_{d-1} z_{r+1}^{d-1} + z_{r+1}^d$$

with $f_i \in \overline{\mathbb{Q}}[z_0, \ldots, z_r]$ homogeneous of degree $d-i$, $f(0, \ldots, 0, 1) \neq 0$ and d the degree of X with respect to $\pi^* \mathcal{O}_{\mathbb{P}^{r+1}}(1)$.

Now let S be the homogeneous coordinate ring of $\pi(X)$, say

$$S = \overline{\mathbb{Q}}[z_0, \ldots, z_{r+1}]/J$$

for some homogeneous ideal J (its generated by f).

Definition 3.5

Let $f \in K[t_1, \ldots, t_n]$ be a polynomial, then the ***height*** of f is defined by

$$h(f) = \sum_{v \in M_K} \log |f|_v$$

where $|f|_v = \max_j |a_j|_v$ where the max is taking over all coefficients of f.

Definition 3.6

Let $\phi : X \to \mathbb{P}^n_{\overline{\mathbb{Q}}}$ be a closed embedding and $x_0, \ldots, x_n$ be the standard coordinates. Let $\mathbf{p}$ be a vector with entries $p_i \in S$ where $i = 0, \ldots, n$, homogeneous of degree d. Then we say $\mathbf{p}$ is a ***presentation*** of ϕ if the following conditions are satisfied:

1. If $l \in \{0, \ldots, n\}$ and $x_l|_X \neq 0$, then $p_l \neq 0$
2. If l is as in (1) and $i \in \{0, \ldots, n\}$, then

$$\frac{p_i}{p_l} = \left.\frac{x_i}{x_l}\right|_X$$

in $\overline{\mathbb{Q}}(X)$

the number d here is called the degree of $\mathbf{p}$ and its denoted by $d(\mathbf{p})$. Note p_i are polynomials, and thus we can define the height $h(\mathbf{p}) := \max_{i=0,\ldots,n} h(p_i)$.

Theorem 3.7

Let $\phi : X \to \mathbb{P}^n$, $\psi : X \to \mathbb{P}^m$ be closed embeddings with corresponding presentations $\mathbf{q}, \mathbf{p}$. We assume $\phi^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \psi^* \mathcal{O}_{\mathbb{P}^m}(1)$ and $d(\mathbf{q}) \geq 1$. Then for $P \in X$, we have

$$h_\phi(P) - h_\psi(P) \leq C_1 (n+1) d(\mathbf{q})^{r^2+r} (A + B + \log(n+1) + C_2)$$

where

$$C_1 = d\frac{(d+1)^r(r+1)^r}{r!}, C_2 = (d-1)h(f) + d(d+r+1) + r + 1$$

$$A = h(\mathbf{p}) + h(\mathbf{q})$$

$$B = r\log\left(6 + \frac{6d(\mathbf{p})}{r}\right) + r\log\left(6 + \frac{6d(\mathbf{q})}{r}\right)$$