1 Recap

Last time we talked about heights on projective spaces.

In particular, for an equivalence class of non-trivial absolute values v on field K, and extension L/K with place $w \mid v$, we defined its normalization by

$$|x|_w^{\trianglelefteq} := |N_{L_w/K_v}(x)|_v^{1/[L:K]}$$

Next, for a set of places M_K, we defined $M_L := \{|x|_w^{\trianglelefteq} : v \in M_K, w \mid v\}$. In particular, for a number field K, we define M_K to be $\{|x|_v^{\trianglelefteq} : v \in M_{\mathbb{Q}}, w \mid v\}$ where $M_{\mathbb{Q}}$ is the usual set of places on $\mathbb{Q}$. For a number field K, M_K satisfies $\prod_{v \in M_K} |x|_v = 1$ for all $x \in K \backslash \{0\}$.

Then, for a point $P \in \mathbb{P}^n_{\overline{\mathbb{Q}}}$, we defined

$$h(P) := \sum_{v \in M_K} \max_j \log |P_j|_v$$

where K is a number field contains all the coordinates of P. This definition is independent of K and action of $\overline{\mathbb{Q}}^{\times}$. In particular, this defines a map $h : \mathbb{P}^n_{\overline{\mathbb{Q}}} \to \mathbb{R}$, and we proved

$$\ker(\overline{\mathbb{Q}} \xrightarrow{h} \mathbb{R}) = \bigcup_{r \geq 1} \mu_r$$

That is, $\xi \in \overline{\mathbb{Q}}$ has height 0 if and only if ξ is a root of unity.

2 Wrap-up Heights on Projective Space

The main thing we will talk about will be Northcott's theorem.

Before that, we mention Segre embedding. The first one is actually very simple. Recall coordinate-wise we have closed immersion

$$\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$

given by

$$((x_0 : \ldots : x_n), (y_0 : \ldots : y_m)) = (x_i y_j)$$

where (i, j) are ordered lexicographically. This in particular implies $h(\mathbf{x} \otimes \mathbf{y}) = h(\mathbf{x}) + h(\mathbf{y})$.

Next, we define $\log^+(x) = \max(0, \log(x))$, and we see for $P \in \mathbb{A}^n$ identified by $(1 : P_1 : \ldots : P_n) \in \mathbb{P}^n$, we have

$$h(P) = \sum_{v \in M_K} \max_j \log^+ |P_j|_v$$

Proposition 2.1

Let $P^1, ..., P^r \in \mathbb{A}^n_{\overline{\mathbb{Q}}}$, then

$$h(\sum P^i) \leq \sum h(P^i) + \log r$$

Proof. We assume $P^i \in \mathbb{A}^n_K$ for some number field K. Then

$$h(\sum_i P^i) = \sum_{v \in M_K} \max_j \log^+ |\sum_i P^i_j|_v$$

If v is non-archimedean, then by strong trig inequality, we see

$$|\sum_i P^i_j|_v \leq \max_k |P^k_j|_v$$

If v is archimedean, then

$$|\sum_i P^i_j|_v \leq |r|_v \cdot \max_k |P^k_j|_v$$

but then $\sum_{v|\infty} \log |r|_v = \log r$. This shows

$$\begin{aligned} h(\sum_i P^i_j) &\leq \log r + \sum_{v \in M_K} \max_{j,k} \log^+ |P^k_j|_v \\ &\leq \log r + \sum_k \sum_{v \in M_K} \max_j \log^+ |P^k_j|_v \end{aligned}$$

Q.E.D.

The next small topic is fundamental inequality.

Lemma 2.2

For $\alpha \in K \backslash \{0\}$ and $\lambda \in \mathbb{Q}$, $h(\alpha^\lambda) = |\lambda| h(\alpha)$. In particular, $h(1/\alpha) = h(\alpha)$.

This follows from $\log |\alpha|_v = \log^+ |\alpha|_v - \log^+ |1/\alpha|_v$, then sum over all places.

Now let $S \subseteq M_K$ be a finite set of places. For $\alpha \in K \backslash \{0\}$, we have

$$\sum_{v \in S} \log |\alpha|_v \leq h(\alpha)$$

If we use $1/\alpha$, the above lemma says

$$\sum_{v \in S} \log |\alpha|_v \geq -h(\alpha)$$

Thus, we see

$$-h(\alpha) \leq \sum_{v \in S} \log |\alpha|_v \leq h(\alpha)$$

Theorem 2.3: Northcott's Theorem

There are only finitely many algebraic numbers of bounded degree and bounded height.

Proof. To make the statement above more precise, we will show the following. For any $B, D \geq 0$, the set

$$\{P \in \mathbb{P}^n_{\overline{\mathbb{Q}}} : H(P) \leq B \text{ and } [\mathbb{Q}(P) : \mathbb{Q}] \leq D\}$$

is finite. In particular, for any fixed number field K, $\{P \in \mathbb{P}^n_k : H(P) \leq B\}$ is finite. In the above, $\mathbb{Q}(P)$ is the minimal number field containing all coordinates of P.

Now let $P = (P_0 : \ldots : P_n)$ where we assume some $P_i = 1$. Then for any absolute value v and index i we have

$$\max(\|P_0\|_v, \ldots, \|P_n\|_v) \geq \max(\|P_i\|_v, 1)$$

Hence, we see

$$H(P) \geq H(P_i)$$

for all $0 \leq i \leq n$. Further, its clear $\mathbb{Q}(P) \supseteq \mathbb{Q}(P_i)$, hence it suffices to prove for each $1 \leq d \leq D$, the set

$$\{x \in \overline{\mathbb{Q}} : H(x) \leq B \text{ and } [\mathbb{Q}(x) : \mathbb{Q}] = d\}$$

is finite.

Let $\xi \in \overline{\mathbb{Q}}$ have degree d and $k = \mathbb{Q}(x)$. We write $\mathbf{x} := (\xi_1, \ldots, \xi_d)$ for the conjugates of ξ over $\mathbb{Q}$, and we let

$$F_\xi(x) = \prod_{j=1}^{d}(x - x_j) = \sum_{r=0}^{d}(-1)^r s_r(\mathbf{x})x^{d-r}$$

the minimal polynomial of x over $\mathbb{Q}$. However, we see

$$\begin{aligned} |s_r(\mathbf{x})|_v &= \left| \sum_{1 \leq i_1 < \ldots < i_r \leq d} \xi_{i_1} \ldots \xi_{i_r} \right|_v \\ &\leq c(v, r, d) \max_{1 \leq i_1 < \ldots < i_r \leq d} |\xi_{i_1} \ldots \xi_{i_r}|_v \\ &\leq c(v, r, d) \max_{1 \leq i \leq d} |\xi_i|_v^r \end{aligned}$$

where $c(v, r, d) = \binom{d}{r} \leq 2^d$ if v is archimedean, and 1 if v is non-archimedean.

Thus we see

$$\max(|s_0(\mathbf{x})|_v, \ldots, |s_d(\mathbf{x})|_v) \leq c(v, d) \prod_{i=1}^{d} \max(|\xi_i|_v, 1)^d$$

where $c(v, d) = 2^d$ if v is archimedean and 1 otherwise.

Now multiply this inequality over all $v \in M_K$, where $K = \mathbb{Q}(x)$, and take $[K:\mathbb{Q}]$th root, we see

$$H(s_0(\mathbf{x}), \dots, s_d(\mathbf{x})) \le 2^d \prod_{i=1}^{d} H(x_i)^d$$

But the x_i's are conjugates, and we know heights are invariant under Galois action, thus $H(x_i)$'s are all equal. This shows

$$H(s_0(\mathbf{x}), \dots, s_d(\mathbf{x})) \le 2^d H(x)^{d^2}$$

Now suppose x is in the set

$$\{x \in \overline{\mathbb{Q}} | H(x) \le B \text{ and } [\mathbb{Q}(x):\mathbb{Q}] = d\}$$

Then we just proven x is the root of a polynomial $F_x(T)$ whose coefficients $s_0, \dots, s_d$ are bounded by $2^d B^{d^2}$. However, it is easy to see $\mathbb{P}^d(\mathbb{Q})$ has only finitely many points of bounded height, so there are only finitely many possibilities for $F_x(T)$, and we are done.

3 Local Heights

Now let X be projective variety, and suppose we want to define a height on X based on the heights on projective space. Then immediately we see this notion must depend on the embeddings we are using.

Thus, in order to define a notion which extends heights on projective space, we must keep track of the morphism $X \to \mathbb{P}^n$. This data is the same as a base-point free line bundle $\mathscr{L}$ on X, together with n sections $s_1, \dots, s_n$ that do not vanish at the same time.

For us, to define local heights, we require more than just this information. Instead, we require a decomposition $\mathscr{Q} = \mathscr{L} \otimes \mathscr{M}^{-1}$ where $\mathscr{L}, \mathscr{M}$ are both base-point free line bundles with a set of generating global sections.

Convention

Throughout this section, we will let K be a field and $|\cdot|$ be a fixed absolute value on $\overline{K}$.

Definition 3.1

Let $D = (U_i, f_i)$ be a Cartier divisor on $X = \bigcup U_i$, and suppose $\mathscr{O}(D) = \mathscr{L} \otimes \mathscr{M}^{-1}$ for base-point free line bundles $\mathscr{L}, \mathscr{M}$. Then a ***presentation*** of D is the data $\mathcal{D} = (s_D, \mathscr{L}, \mathbf{s}, \mathscr{M}, \mathbf{t})$ where $\mathbf{s}, \mathbf{t}$ are generating global sections of $\mathscr{L}, \mathscr{M}$, respectively, and s_D is the meromorphic section associated with $\mathcal{O}(D)$.

We note for Cartier divisor D, to get the line bundles $\mathcal{L}, \mathcal{M}$ as above, we take $D_1 = mH + D$ and $D_2 = mH$, where H is a very ample divisor on X. Then for m large enough we see D_1, D_2 are ample and $D_1 - D_2 \equiv D$.

Proposition 3.2

For $P \notin \mathrm{supp}(D)$, we define the ***local height*** *(wrt $\mathcal{D}$) to be*

$$\lambda_{\mathcal{D}}(P) = \max_k \min_l \log |\frac{s_k}{t_l s_D}(P)|$$

Example 3.3

Let f be non-zero rational function on X with Cartier divisor $D = D(f)$. Then $\mathcal{O}(D) = \mathcal{O}_X$ and f is a meromorphic section of $\mathcal{O}(D)$. Thus there is a local height λ_f associated with presentation $(f, \mathcal{O}_X, 1, \mathcal{O}_X, 1)$. For $P \notin \mathrm{supp}(D)$, we have

$$\lambda_f(P) = -\log |f(P)|$$

In particular, if g is another non-zero rational function on X, then $\lambda_{fg} = \lambda_f + \lambda_g$ and $\lambda_{f^{-1}} = -\lambda_f$.

Next, we have two constructions for $\lambda_{\mathcal{D}}$.

Addition/Negation

Let $\mathcal{D}_1, \mathcal{D}_2$ be two presentations of D_1, D_2, respectively. Then we can define a presentation $\mathcal{D}$ of $D_1 + D_2$ as follows:

$$(s_{D_1} s_{D_2}, \mathcal{L}_1 \otimes \mathcal{L}_2, (s_{1k} s_{2k'})_{k,k'}, \mathcal{M}_1 \otimes \mathcal{M}_2, (t_{1l} t_{2l'})_{l,l'})$$

Thus, we can define $\lambda_{\mathcal{D}_1} + \lambda_{\mathcal{D}_2}$ be the local height associated with $\mathcal{D}$ as above.

Next, for $\mathcal{D} = (s_D, \mathcal{L}, \mathbf{s}, \mathcal{M}, \mathbf{t})$ we can define a presentation for $-D$ by

$$(s_D^{-1}, \mathcal{M}, \mathbf{t}, \mathcal{L}, \mathbf{s})$$

and this two operations together makes the set of $\lambda_{\mathcal{D}}$ into a monoid.

Pullback

Now let $\pi : Y \to X$ be dominant morphism of irreducible projective varieties over K. Let $\mathcal{D} = (s_D, \mathcal{L}, \mathbf{s}, \mathcal{M}, \mathbf{t})$, then we can define a presentation $\pi^*\mathcal{D}$ by

$$(\pi^* s_D, \pi^* \mathcal{L}, \pi^* \mathbf{s}, \pi^* \mathcal{M}, \pi^* \mathbf{t})$$

In particular, we see $\lambda_{\pi^*\mathcal{D}}(P) = \lambda_{\mathcal{D}}(\pi(P))$ for well-defined P, i.e. $P \in Y, \pi(P) \notin \mathrm{supp}(D)$.