1 Results From Before

Lemma 1.1: Constancy Lemma

Let X, Y, Z be varieties such that X is complete and X, Y geometrically irreducible. If $f : X \times Y \to Z$ is a morphism such that $f(X \times \{y_0\}) = \{z_0\}$ for some $y_0 \in Y$ and $z_0 \in Z$, then $f(X \times \{y\})$ is a point for all $y \in Y$.

Corollary 1.1.1

Let $\phi : A \to G$ be a morphism of abelian variety A into group variety G. Then the map

$$\phi : A \to G, \quad a \mapsto \phi(a)\phi(\epsilon_A)^{-1}$$

is a homomorphism of group varieties.

Proposition 1.2

For a group variety G over K, the following are equivalent:

1. *G is connected*
2. *G is geometrically connected*
3. *G is irreducible*
4. *G is geometrically irreducible*

In particular, a connected complete geometrically reduced group variety over K is abelian variety.

2 The Picard Variety

Elliptic curves are the only standard explicit examples of abelian varieties. This is because higher-dimensional abelian varieties can be defined only by means of a very large number of equations, and little can be understood by just looking at the equations.

However, abelian varieties are ubiquitous in algebraic geometry and they occur most naturally through the Picard variety, which we will study here.

Let us fix a ground field K and algebraic closure $\overline{K}$.

If $\phi : X \to Y$ is a morphism of varieties over K and $y \in Y$, then the fiber of ϕ over y is denoted by $X_y = X \times_Y \kappa(y)$. The pullback of $\mathscr{L} \in \mathrm{Pic}(X)$ to the fiber X_y is denoted $\mathscr{L}_y$. Its an element of $\mathrm{Pic}(X_y)$. Note X_y and $\mathscr{L}_y$ are only defined over $\kappa(y)$. For a fixed y we often identify X with X_y using the map $x \mapsto (x, y)$, which is only defined over $\kappa(y)$.

In the following, we consider $\mathscr{L} \in \mathrm{Pic}(X \times Y)$ and the fibers with respect to the projections p_1, p_2 onto the factors. For $x \in X, y \in Y$, we have

$$\mathscr{L}_y = \mathscr{L}|_{X\times\{y\}} \in \mathrm{Pic}(X_{\kappa(y)}), \quad \mathscr{L}_x = \mathscr{L}|_{\{x\}\times Y} \in \mathrm{Pic}(Y_{\kappa(x)})$$

Theorem 2.1: Seesaw Principle

Let X be a geometrically irreducible smooth complete variety over K and Y an irreducible smooth variety over K. Let $\mathscr{L} \in \mathrm{Pic}(X \times Y)$ and suppose there is dense open $U \subseteq Y$ so $\mathscr{L}_y = 0$ for all $y \in U$. Then $\mathscr{L}$ is equal to the pullback of an element of $\mathrm{Pic}(Y)$ by p_2.

This result holds even without the smoothness assumption. We often use this principle in the following form.

Corollary 2.1.1

Let X, Y be smooth varieties over K and assume Y is irreducible and that X is complete and geometrically irreducible. Let $\mathscr{L} \in \mathrm{Pic}(X \times Y)$ with $\mathscr{L}_y = 0$ for all y in an open dense subset of Y and with $\mathscr{L}_x = 0$ for all $x \in X(K)$. Then $\mathscr{L} = 0$.

Proof. By Theorem 2.1, we have $\mathscr{L} = p_2^*\mathscr{L}'$ for some $\mathscr{L}' \in \mathrm{Pic}(Y)$. Now consider the closed embedding $\iota_x : Y \to X \times Y$, $y \mapsto (x, y)$. Since $p_2 \circ \iota_x$ is the identity map on Y, we see

$$\mathscr{L}' = \iota_x^* p_2^* \mathscr{L}' = \mathscr{L}_x = 0$$

Since this holds for all x, we are done.

Corollary 2.1.2

Let A be abelian variety over K, p_i the ith projection $A \times A$ onto A, and m be addition as usual. The following are equivalent for $\mathscr{L} \in \mathrm{Pic}(A)$:

1. $m^*(\mathscr{L}) = p_1^*\mathscr{L} + p_2^*\mathscr{L}$
2. $\tau_a^*(\mathscr{L}) = \mathscr{L}$ *for all* $a \in A$

If (1) *or* (2) *holds, then* $[-1]^*(\mathscr{L}) = -\mathscr{L}$.

Proof. The equivalence is a consequence of

$$(m^*(\mathscr{L}) - p_1^*(\mathscr{L}) - p_2^*(\mathscr{L}))|_{A\times\{a\}} = \tau_a^*(\mathscr{L}) - \mathscr{L}$$

and the seesaw principle from Corollary 2.1.1. If we pullback equation in (1) by the morphism

$$A \to A \times A, \quad a \mapsto (a, -a)$$

then we get $[-1]^*(\mathscr{L}) = -\mathscr{L}$.

Theorem 2.2

Let X be an irreducible smooth complete variety over K and $P_0 \in X(K)$ a base point of X. Then the group $\mathrm{Pic}^0(X_{\overline{K}})$ has a unique structure as an abelian variety over K, called the Picard variety and denoted by $\mathcal{P}\mathrm{ic}^0(X)$, with the properties:

1. *There is $\mathscr{P} \in \mathrm{Pic}(X \times \mathcal{P}\mathrm{ic}^0(X))$ such that $\mathscr{P}_{\mathscr{B}} = \mathscr{B}$ for $\mathscr{B} \in \mathcal{P}\mathrm{ic}^0(X)$ and $\mathscr{P}_{P_0}$ is trivial*
2. *For any subfamily $\mathscr{L}$ of $\mathrm{Pic}^0(X)$ parametrized by an irreducible variety T over K, the set-theoretic map*

$$T \to \mathcal{P}\mathrm{ic}^0(X), \quad t \mapsto \mathscr{L}_t$$

is actually a morphism over K.

The uniquely determined class $\mathscr{P}$ is called the ***Poincaré class****.*

Now given $\phi : X \to X'$ a pointed morphism between complete smooth variety over K with base point $P_0 \in X(K)$ and $P_0' \in X'(K)$ respectively (i.e. we require $\phi(P_0) = P_0'$). Then the map

$$\hat{\phi} : \mathcal{P}\mathrm{ic}^0(X') \to \mathcal{P}\mathrm{ic}(X), \quad \mathscr{L}' \mapsto \phi^*\mathscr{L}'$$

is called the dual map of ϕ, which is a homomorphism of abelian varieties.

Remark 2.3: Fact

In particular, if F/K is a field extension, then:

1. by base change, $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_F)$
2. $\mathcal{P}\mathrm{ic}^0(X_F) = \mathcal{P}\mathrm{ic}^0(X)_F$ and its Poincaré class is obtained from $\mathscr{P}$ by base change of F
3. $\mathcal{P}\mathrm{ic}^0(X)(F) = \mathrm{Pic}^0(X_F)$ by identifying b with $\mathscr{P}_b$

Remark 2.4

By the seesaw principle as in Corollary 2.1.1 and Corollary 2.3, the Poincaré class $\mathscr{P}$ is uniquely characterized by the conditions:

1. $\mathscr{P}_{\mathscr{L}} = \mathscr{L}$ for any $\mathscr{L} \in \mathcal{P}\mathrm{ic}^0(X)$, i.e. the fiber of $\mathscr{P}$ at any degree 0 line bundle is just that line bundle itself
2. $\mathscr{P}_{P_0} = 0$

Remark 2.5

In the complex analytic situation, take X be irreducible proper smooth complex variety viewed as a compact connected complex manifold. View the transition functions $(g_{\alpha,\beta})$ for a line bundle $\mathscr{L}$ on X as a Čech cocycle valued in $\mathscr{O}_X^\times$, we see $\mathrm{Pic}(X) = H^1(X, \mathscr{O}_X^\times)$. Now consider the exponential map short exact sequence

$$0 \to \mathbb{Z}_X \to \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \to 0$$

Now take cohomology long sequence, we get

$$\begin{array}{l} 0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \\ \to H^1(X,\mathbb{Z}) \to H^1(X,\mathscr{O}_X) \to H^1(X,\mathscr{O}_X^\times) = \mathrm{Pic}(X) \\ \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to \dots \end{array}$$

where c_1 gives the first Chern class of line bundles. A line bundle is equivalent to 0 if and only if $\mathscr{L}$ has (first) Chern class 0. If we use canonical isomorphism

$$H^1(X, \mathscr{O}_X) \cong H^{0,1}(X)$$

from Dolbeault complex, we conclude the Picard variety is biholomorphic to the complex torus $H^{0,1}(X)/H^1(X, \mathbb{Z})$.

3 The Theorem of Square

In the above, we defined the Picard variety $\mathcal{P}\mathrm{ic}^0(X)$ of irreducible smooth complete variety X with K-rational base point. In this section, we assume X is abelian variety over K and the base point is the origin.

Definition 3.1

Let A be smooth complete abelian variety, then $\mathcal{P}\mathrm{ic}^0(A)$ is called the ***dual abelian variety*** of A and denoted by $\hat{A}$.

The theorem of square says for any $\mathscr{L} \in \mathrm{Pic}(A)$, the point $\phi_{\mathscr{L}}(a) := \tau_a^*(\mathscr{L}) - \mathscr{L}$ is in $\hat{A}$ and additive in $a \in A$.

As a consequence of the theorem of the square, we will prove:

1. abelian variety is always projective

2. for $\mathscr{L}$ ample, $\phi_{\mathscr{L}}$ is surjective with finite kernel (in particular $\dim \hat{A} = \dim A$)

Theorem 3.2

Let $\mathscr{L} \in \mathrm{Pic}(A)$ and $a \in A$. Then $\phi_{\mathscr{L}}(a) := \tau_a^*(\mathscr{L}) - \mathscr{L} \in \mathcal{P}\mathrm{ic}^0(A)(\kappa(a))$ and $\phi_{\mathscr{L}} : A \to \mathcal{P}\mathrm{ic}^0(A)$ is a homomorphism of abelian varieties over K.

Proof. Let p_i be the ith projection of $A \times A$ onto A and consider

$$\mathscr{L}' = m^*(\mathscr{L}) - p_1^*(\mathscr{L}) - p_2^*(\mathscr{L})$$

on $A \times A$. We already remarked in the proof of Corollary 2.1.2 that

$$\mathscr{L}'|_{A \times \{a\}} = \tau_a^*(\mathscr{L}) - \mathscr{L}$$

for $a \in A$. Thus $\phi_{\mathscr{L}}(a) \in \mathrm{Pic}^0(A_{\kappa(a)}) = \mathcal{P}\mathrm{ic}^0(A)(\kappa(a))$ by the definition of algebraic equivalence and Corollary 2.3. Since $\mathscr{L}'|_{\{0\} \times A} = 0$, $\mathscr{L}'$ is a subfamily of $\mathrm{Pic}^0(A)$ parametrized by A. Theorem 2.2 shows $\phi_{\mathscr{L}}$ is a morphism of varieties defined over K. Since $\phi_{\mathscr{L}}(0)$ is trivial, the map is a homomorphism of abelian varieties (Corollary 1.1.1).

Q.E.D.

Theorem 3.3: Theorem of Square

For $a, b \in A$, we have

$$\tau_{a+b}^*(\mathscr{L}) + \mathscr{L} = \tau_a^*(\mathscr{L}) + \tau_b^*(\mathscr{L})$$

Proof. Apply Theorem 3.2, then subtract $2\mathscr{L}$ on both side.

Q.E.D.

Theorem 3.4

Let $\mathscr{B} \in \mathrm{Pic}(A)$ such that $\phi_{\mathscr{B}} = 0$. Then for any ample $\mathscr{L} \in \mathrm{Pic}(A)$, there is some $a \in A$ with

$$\mathscr{B} = \tau_a^*(\mathscr{L}) - \mathscr{L}$$

Remark 3.5

The kernel of $\phi_{\mathscr{L}}$ gives much information about $\mathscr{L}$. If $\mathscr{L}$ is ample, then the kernel is finite. We will prove a partial converse of this statement, which we will use later. On the other hand, $\ker(\phi_{\mathscr{L}}) = A$ if $\mathscr{L} \in \mathrm{Pic}^0(A)$. These statements about kernel will be proved next.

Fact 3.6: Ample $\mathscr{L} \cong \mathscr{O}_X$ means X affine

We first recall the following result. Let X be qcqs scheme, the following are equivalent:

1. X is quasi-affine
2. There is line bundle $\mathscr{L}$ such that $\mathscr{L}$ and $\mathscr{L}^{-1}$ are ample
3. Every quasi-coherent $\mathscr{O}_X$-module is generated by its global sections
4. The canonical morphism $X \to \operatorname{Spec}(\Gamma(X, \mathscr{O}_X))$ is quasi-compact open schematically dominant immersion.

Now, we claim if X proper variety over k, $\mathscr{L}$ ample line bundle with $\mathscr{L} \cong \mathscr{O}_X$, then X is affine. To see this, by assumption and the result above we see $X \to \operatorname{Spec}\Gamma(X, \mathscr{O}_X)$ is open immersion. Now X is also proper, which means $X \to \operatorname{Spec}\Gamma(X, \mathscr{O}_X)$ is closed. Thus X must be affine as desired. In particular, note X proper and affine means X is finite.

Proposition 3.7

A class $\mathscr{L} \in \operatorname{Pic}(A)$ is ample if and only if $\ker(\phi_{\mathscr{L}})$ is finite and $H^0(A, \mathscr{L}^n) \neq 0$ for some $n > 0$.

Proof. Assume $\mathscr{L}$ is ample. Let B be the connected component of the closed subgroup $\ker(\phi_{\mathscr{L}})$ containing 0. For $b \in B$ we have

$$\tau_b^* \mathscr{L} = \mathscr{L}$$

and hence

$$[-1]^*(\mathscr{L}|_B) = -\mathscr{L}|_B$$

by Corollary 2.1.2. Since

$$0_B = \mathscr{L}|_B + [-1]^*(\mathscr{L}|_B)$$

is ample, B has to be the trivial abelian subvariety $\{0\}$ (using the fact A is complete and then Fact 3.6, and the fact the only abelian affine group variety is $\mathbb{A}^0$). Thus $\ker(\phi_{\mathscr{L}})$ is finite. Choose n so large that $\mathscr{L}^n$ is very ample, which gives $H^0(A, \mathscr{L}^n) \neq 0$.

In the other direction, we may assume $H^0(A, \mathscr{L}) \neq 0$, i.e. there is an effective divisor D so $\mathscr{O}(D) \cong \mathscr{L}$. Thus Lemma 3.8 shows $\mathscr{L}$ is ample.

Q.E.D.

Lemma 3.8

Let D be effective divisor on A and suppose the subgroup $\{a \in A : \tau_a^*(D) = D\}$ is finite. Then D is ample on A.

Let us omit this proof during the talk if we dont have time.

Proof. Note D is ample iff $D_{\overline{K}}$ is ample over $A_{\overline{K}}$. Thus we assume K is ACF. The proof then proceeds by proving first the linear system $|2D|$ is base-point free and define a morphism ϕ of A into some projective space. Then we show ϕ is finite morphism and the conclusion comes by pullback. The details are as follows.

Let $a, b \in A$. If b is in the support of the effective divisor

$$E_a := \tau_a^*(D) + \tau_{-a}^*(D)$$

then $a + b$ or $b - a$ is in the support of D. For any given $b \in A$ we can always find $a \notin (D - b) \cup (b - D)$, i.e. $b \notin \mathrm{supp}(E_a)$. Then by the theorem of the square 3.3 the effective divisor E_a is an element of $|2D|$. Thus the linear system $|2D|$ is base-point free and thus defines a morphism $\phi : A \to \mathbb{P}_K^n$.

The morphism ϕ is proper. Let F be an irreducible component of any fiber. All elements of $|2D|$ are pullbacks of hyperplanes by the definition of ϕ. Now for any $a \in A$ either F is contained in the support of E_a or $F \cap \mathrm{supp}(E_a) = \emptyset$, hence we can find $a \in A$ so F and the support of E_a are disjoint, i.e. $a \notin \mathrm{supp}(D) - F$. Let Z be an irreducible component of D, then $Z - F$ is irreducible closed subset of A not containing a. We conclude $Z - F$ is of codimension 1. Now note for any $b \in F$, we have

$$Z - F = Z - b$$

whence it follows Z is invariant by translation in $F - F$. Therefore, the same is true for D instead of Z. By assumption, this is only possible for $\dim(F) = 0$ and we conclude ϕ has finite fiber. Thus, since ϕ is proper, it must also be finite (finite fiber=quasi-fintie, proper+quasi-finite means finite). Now recall pullback of ample by finite morphism is ample, we see $2D$ is ample.

Q.E.D.

Corollary 3.8.1

An abelian variety is projective.

Proof. Let U be affine open subset of A containing 0. We may assume $\dim(A) \geq 1$. Let $Z_1, \ldots, Z_r$ be irreducible components of $A \backslash U$. Enlarging them, we may assume $Z_1, \ldots, Z_r$ are prime divisors. In order to see this note the complement of a divisor in an affine smooth variety is smooth. Setting

$$D = \sum Z_i$$

the subgroup $B = \{a \in A : \tau_a^*(D) = D\}$ is closed and for $b \in B$, $U + b = B$. Since $0 \in U$, we have

$$B \subseteq U$$

As a complete variety, B must be finite. Lemma 3.8 shows D is ample, hence A is projective.

Q.E.D.

Proposition 3.9

For $\mathscr{B} \in \operatorname{Pic}(A)$, the following are equivalent:

1. $\mathscr{B} \in \operatorname{Pic}^0(A)$
2. $\ker(\phi_{\mathscr{B}}) = A$
3. *For every ample $\mathscr{L} \in \operatorname{Pic}(A)$, there is $a \in A$ so $\mathscr{B} = \tau_a^*(\mathscr{L}) - \mathscr{L}$*
4. *There is ample $\mathscr{L} \in \operatorname{Pic}(A)$, such that $\mathscr{B} = \tau_a^*(\mathscr{L}) - \mathscr{L}$ for some $a \in A$*

Proof. $(1) \Rightarrow (2)$: By Corollary 2.3, we may assume K is ACF. Let

$$\phi : A \to \mathcal{P}\mathrm{ic}^0(A) \to \mathcal{P}\mathrm{ic}^0(A)$$

be the map given by $(a, \mathscr{B}) \mapsto \tau_a^* \mathscr{B}$. We will prove below this is a morphism. For $T = A \times \mathcal{P}\mathrm{ic}^0(A)$, consider

$$\mathscr{L} := (m \times \mathrm{Id}_{\mathcal{P}\mathrm{ic}^0(A)})^*(\mathscr{P}) \in \operatorname{Pic}(A \times T)$$

where m denotes the addition morphism as usual. Note the restriction of $m \times \mathrm{Id}_{\mathcal{P}\mathrm{ic}^0(A)}$ to $A \times \{a\} \times \{\mathscr{B}\}$ is given by $\tau_a \times \{\mathscr{B}\}$, by identifying $A \times \{a\} \times \{\mathscr{B}\}$ with A. By Remark 2.4 and the rule $(f \circ g)^* = g^* \circ f^*$ we get

$$\mathscr{L}|_{A \times \{a\} \times \{\mathscr{B}\}} = \tau_a^* \mathscr{B}$$

and similarly

$$\mathscr{L}|_{\{0\} \times T} = \mathscr{P}$$

Let us denote by p_2 the projection of $A \times T$ onto T. The subfamily $\mathscr{L} - p_2^* \mathscr{P}$ of $\operatorname{Pic}^0(A)$ parametrized by T induces a morphism $T \to \mathcal{P}\mathrm{ic}^0(A)$, which is equal to ϕ (Theorem 2.2). Since $\phi(A \times \{0\}) = 0$, the constancy lemma 1.1 shows $\tau_a^*(\mathscr{B}) = \mathscr{B}$ for all $a \in A$, which proves the claim.

$(2) \Rightarrow (3)$: This is Theorem 3.4.

Clearly $(3) \Rightarrow (4)$ as the existence of an ample class is by Corollary 3.8.1.

$(4) \Rightarrow (1)$: Theorem 3.2.

Q.E.D.

Definition 3.10

The Picard variety $\mathcal{P}\mathrm{ic}^0(A)$ is called the ***dual abelian variety*** of A and will be denoted by $\hat{A}$.

Corollary 3.10.1

The dual abelian variety of A has the same dimension as A.

Proof. There is an ample $\mathscr{L} \in \mathrm{Pic}(A)$, and thus $\phi_{\mathscr{L}} : A \to \hat{A}$ is surjective and finite. Thus they have the same dimension.