Goal:

1. define group varieties
2. constancy lemma
3. generic property holds everywhere on group variety, hence rational map to abelian variety is always morphism at smooth points

In this section we will always work over field K with a choice of algebraic closure $\overline{K}$. All objects will be over K.

1 Basics

Definition 1.1

Let S be a scheme and G an S-scheme, then we say G is a ***group scheme*** if there is a factorization of the functor $h_G := \mathrm{Hom}(-, G)$ from $(\mathbf{Sch}/S)^{\mathrm{opp}}$ to $(\mathbf{Set})$ through the forgetful functor $(\mathbf{Grp}) \to (\mathbf{Set})$.

By Yoneda's lemma, we see the above is the same as the following two set of data:

1. For all S-scheme T, there is a group structure on $h_G(T) = \mathrm{Hom}_S(T, G)$ which is functorial in T
2. Three S-morphisms $m : G \times_S G \to G$, $i : G \to G$ and $e : S \to G$, which correspond to multplication, inverse and unit of the group

If G happens to be a variety, then we say G is a group variety (gv for short).

Definition 1.2

An ***abelian variety*** (av for short) is a geometrically irreducible and geometrically reduced complex group variety.

Example 1.3

M_n, the set of n by n matrices over K, is irreducible affine group variety. The determinant gives $\det : M_n \to \mathbb{A}^1_K$, and thus we see $\mathrm{GL}(n)$, the complement of the vanishing of det, is affine open irreducible subvariety of M_n. In particular, $\mathrm{SL}(n)$, defined by $\det(a) = 1$, is a subvariety thats also affine group variety.

Some irrelevant facts:

1. Every affine group variety is isomorphic to some closed subgroup of $\mathrm{GL}(n)$.
2. Let G be irreducible group variety over pefect field K, then there is a smallest irreducible affine closed subgroup H and abelian variety A so

$$0 \to H \to G \to A \to 0$$

In particular, we see to study general gv, it suffices to understand both affine gv and av. In particular, since the trivial gv $\mathbb{A}^0_K$ is the only complete geometrically irreducible affine variety, there are no other affine group variety that is also av.

Remark 1.4: Don't have to include

Let X be proper irreducible over K and suppose $X \to Y$ is a K-morphism with Y affine of finite type, then this morphism must be constant. To see this, note we may assume both X, Y are reduced by passing to their underlying reduced subscheme. Say $Y = \operatorname{Spec} A$, i.e. $X \to Y$ is the same as $A \to \Gamma(X, \mathscr{O}_X)$. Now since X is proper hence complete, $\Gamma(X, \mathscr{O}_X)$ is finite dimensional K-vector space (to see this, X reduced means $\Gamma(X, \mathscr{O}_X)$ is reduced finite dimensional K-algebra, but X is also irreducible, thus it must be a field). Hence the image of A in $\Gamma(X, \mathscr{O}_X)$ must be a field, say k, then we see $X \to Y$ factor through $X \to \operatorname{Spec} k \to Y$ as desired.

The following lemma roughly says if ϕ is constant on $X \times Y$ on one fiber, then ϕ is constant on all fibers.

Lemma 1.5: Constancy Lemma

Let X, Y, Z be varieties such that X is complete and X, Y geometrically irreducible. If $f : X \times Y \to Z$ is a morphism such that $f(X \times \{y_0\}) = \{z_0\}$ for some $y_0 \in Y$ and $z_0 \in Z$, then $f(X \times \{y\})$ is a point for all $y \in Y$.

Proof. By base change, we may assume $K = \overline{K}$ is ACF. Let U be open affine around z_0. The image

$$C = \{y \in Y : \exists x \in X, f(x, y) \in Z \backslash U\}$$

of $f^{-1}(Z \backslash U)$ by the projection $X \times Y \to Y$ is closed as X is complete. Then

$$V = Y \backslash C = \{y \in Y : \forall x \in X, f(x, y) \in U\}$$

is open neighbourhood of y_0 and, for any $y \in V$, we have $X \to U$, given by $x \mapsto f(x, y)$. Since X is complete and irreducible and U is affine, the morphism has to be constant for any $y \in V$, with image $f(x_0, y)$ choice of a point $x_0 \in X$ (Remark 1.4 above). Now note

$$S = \{y \in Y : |f(X \times \{y\})| = 1\} = \bigcap_{x_1, x_2 \in X} \{y \in Y : f(x_1, y) = f(x_2, y)\}$$

is closed in Y. Since it contains the non-empty open subset V of Y and since Y is irreducible, we conclude $S = Y$, proving our claim.

Q.E.D.

Corollary 1.5.1

Let X, Y be geometrically irreducible variety with at least one K-rational point. We assume X is complete. A morphism $f : X \times Y \to G$ of a product into a group variety *factorizes as $f(x,y) = g(x)h(y)$, for suitable morphism $g : X \to G$ and $h : Y \to G$.*

Proof. We choose $y_0 \in Y(K)$ and define $g : X \to G$ by $g(x) = f(x, y_0)$. The morphism $F : X \times Y \to G$ defined by $F(x,y)g(x)^{-1}f(x,y)$ satisfies $F(X \times \{y_0\}) = \{\epsilon\}$ where $\epsilon \in G$ is the identity of G. Now Constancy Lemma 1.5 shows $F(X \times \{y\})$ is a point, say $h(y)$, for every $y \in Y$, and $f(x,y) = g(x)h(y)$. In order to verify h is a K-morphism, note $h = f(x_0, \cdot)g(x_0)^{-1}$ for any $x_0 \in X(K)$.

Corollary 1.5.2

Let $\phi : A \to G$ be a morphism of abelian variety A into group variety G. Then the map

$$\phi : A \to G, \quad a \mapsto \phi(a)\phi(\epsilon_A)^{-1}$$

is a homomorphism of group varieties.

Proof. Apply the Constancy lemma 1.5 with $f : A \times A \to G$, given by

$$(x,y) \mapsto \psi(x)\psi(y)\psi(xy)^{-1}$$

and with y_0, z_0 the identity of A, G, respectively. We conclude the restriction of f to $A \times \{y\}$ is a constant map for every y. Since $f(\{\epsilon_A\} \times A) = \{\epsilon_G\}$, we deduce f is constant, with image the identity of G.

Corollary 1.5.3

An abelian variety is commutative.

Proof. By Corollary 1.5.2, the inverse map ι is a homomorphism of group varieties. This is equivalent to commutativity.

Example 1.6

The affine line $\mathbb{A}^1_K$ is not complete because $xy = 1$ is closed subvariety of $\mathbb{A}^1 \times \mathbb{A}^1$, while its projection on the second factor is $\mathbb{A}^1 \backslash \{0\}$, not closed in $\mathbb{A}^1_K$. Now consider the morphism $f : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ given by $(x,y) \mapsto xy$. Then f satisfies the hypothesis of Lemma 1.5. This shows constancy lemma does not hold for non-complete X.

From now on, we will use additive notation for abelian varieties, i.e. $m(x,y) = x+y$, $i(x) = -x$, and denote the identity by 0. For $a \in A$, we have the translation map $\tau_a(x) = x+a$. For $n \in \mathbb{Z}$, we denote $[n]$ the endomorphism of A, which is multplication by n. The kernel of $[n]$ is denoted by $A[n]$, and it forms the torsion subgroups of A.

Remark 1.7

In the following, we note if X is locally of finite type and geometrically reduced scheme over K, then X contains an open dense locus of smooth points.

This problem is local on X, thus assume X is quasi-compact with irreducible components X_i. Then $Z = \bigcup_{i \neq j} X_i \cap X_j$ is nowhere dense, and thus we may replace X by $X \backslash Z$. Thus we may assume X is irreducible as $X \backslash Z$ is disjoint union of irreducible schemes. Since X is irreducible and reduced, its integral (integral=irreducible+reduced). Let $\eta \in X$ be its generic point. Then the function field $K(X) = \kappa(\eta)$ is geometrically reduced over K, hence separable over K. Let $U = \operatorname{Spec} A \subseteq X$ be any nonempty affine open so $\kappa(\eta) = A_{(0)}$ is the fraction field of A. This implies (recall the following: if S is finite type k-algebra, $\mathfrak{p} \in \operatorname{Spec} S$ and $\kappa(\mathfrak{p})$ separable over k, then S is smooth at $\mathfrak{p}$ over k if and only if $S_{\mathfrak{p}}$ is regular) A is smooth at (0) over K. By definition this means some principal localization of A is smooth over K.

Proposition 1.8

A geometrically reduced group variety is smooth.

Proof. By base change, we may assume K is ACF. The set of smooth points of X is open, and since X is geometrically reduced, the smooth locus is dense (Remark 1.7). As above, we can define left and right translation by a point of the group variety. They are automorphisms and so the left translation of U is also smooth. If we vary the left translations, then we get an open cover of the group variety, proving the claim.

Q.E.D.

Remark 1.9

Suppose X is K-scheme, then clearly it is geometrically connected implies X is connected.

Next, suppose X is of finite type and connected over K. Then we show if X admits a K-rational point then X is geometrically connected. First, we see if X is quasi-compact, then X is geometrically connected if and only if $X_{K'}$ is connected for all finite separable extension K' of K. We will assume this fact.

Now, if K'/K is finite separable, then we see $\operatorname{Spec} K' \to \operatorname{Spec} K$ is finite flat, and hence universally closed and universally open at the same time. Thus $X_{K'} \to X_K = X$ is open and closed, finite and flat. This means any connected component of $X_{K'}$ surjects onto connected components of X (say Z is a connected component of $X_{K'}$, then $Z \hookrightarrow X_{K'}$ is open and closed, thus $Z \hookrightarrow X_{K'} \to X$ is open and closed, thus the image of Z is open and closed in X, hence a connected component of X).

To conclude the proof, note we assumed X is connected, thus every connected component surjects onto X, which means all connected components have the same K-rational point $x : \operatorname{Spec} K \to X$ in their image. But the base change of this rational point $x_{K'} : \operatorname{Spec} K' \to X$ along $\operatorname{Spec} K' \to \operatorname{Spec} K$ is just a single K'-rational point, thus all the connected components of $X_{K'}$ meet at this single K'-rational point, i.e. $X_{K'}$ is connected.

Proposition 1.10

For a group variety G over K, the following are equivalent:

1. *G is connected*
2. *G is geometrically connected*
3. *G is irreducible*
4. *G is geometrically irreducible*

In particular, a connected complete geometrically reduced group variety over K is abelian variety.

Proof. First, we note K-variety with at least one K-rational point is connected iff its geometrically connected (Remark 1.9). Thus (1) $\iff$ (2). Every irreducible variety is connected, so it remains to prove (2) $\Rightarrow$ (4). We may assume K is ACF and G connected. By Proposition 1.8 shows G is smooth and thus its disjoint union of its irreducible components, i.e. G is irreducible.

Next, as you would guess, we want to study $\operatorname{im}(\phi)$ and $\ker(\phi)$ for $\phi : G \to H$ a homomorphism of group varieties. It can be shown (e.g. you can find this result in SGA) $\operatorname{im}(\phi)$ is a closed subgroup variety of H, but the kernel need more care. To be exact, it will always be a scheme, but its possible to have non-reduced structure, e.g. take $G = H = \mathbb{G}_m$ and $\phi(t) = t^2$, then $\ker(\phi) = \operatorname{Spec} k[t]/(t^2 - 1)$, where $k[t]/(t^2-1)$ is not a integral domain. However, since all our main results will only concern varieties, we will take

$$\ker(\phi) := \{x \in G(\overline{K}) : \phi(x) = \epsilon_H\}$$

which will be a closed subgroup variety of G.

Theorem 1.11: Dimension Theorem

Let $\phi : G \to H$ be a surjective homomorhpism of irreducible group varieties. Then

$$\dim(G) = \dim(H) + \dim(\ker(\phi))$$

This roughly follows from the following: if Y is Noetherian and universally catenary, $f : X \to Y$ surjective morphism of irreducible schemes of finite type, then

$$\dim X = \dim Y + \dim f^{-1}(\eta)$$

where η is the generic point of Y. Using this, and note all fibers of $\phi : G \to H$ are isomorphic to $\ker(\phi)$, we are done.

Lemma 1.12

Let R be Noetherian integral domain, A finitely generated R-algebra, and M a finitely generated A-module. Then there is $s \in R \backslash \{0\}$ such that the localization M_s is free R_s-module.

Theorem 1.13: Generic Flatness

Let $f : X \to Y$ be quasi-compact morphism locally of finite presentation and assume Y is integral. Let $\mathscr{F}$ be quasi-coherent $\mathscr{O}_X$-module of finite presentation. Then there is open dense $U \subseteq Y$ such that $\mathscr{F}|_{f^{-1}(U)}$ is flat over U.

Proof. The question is local on Y, so we assume $Y = \operatorname{Spec} A$ is affine, where A is integral domain. Since f is quasi-compact, we find open affine fintie cover $X = \bigcup_i U_i$. If we find dense open subsets U of Y as in the theorem for each $U_i \to Y$, then their intersection wil lsatisfy the desired conclusion for f.

Thus we may assume $X = \operatorname{Spec} B$ is affine, and then B is A-algebra of finite presentation, and $\mathscr{F}$ is quasi-coherent $\mathscr{O}_X$-module associated with the B-module $M = \Gamma(X, \mathscr{F})$ of finite presentation. By elimination of Noetherianness, we may assume the situation arises by base change for $A_0 \to A$, where A_0 is a Noetherian subring of A, from an analogous situation over A_0. Over A_0, the conclusion follows from Lemma 1.12, and since flatness is stable under base change, we are done.

Q.E.D.

Corollary 1.13.1

Let $f : X \to Y$ be a morphism of finite type and locally of finite presentation, and assume Y is integral. Then there is dense open $U \subseteq Y$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is flat.

Proposition 1.14

Let $\phi : G \to H$ be surjective homomorphism of irreducible group varieties. Then ϕ is flat. Moreover, if $\dim(G) = \dim(H)$, then ϕ is finite and $|\ker(\phi)|$ is equal the separable degree of the field extension $K(G)$ over $K(H)$.

Proof. By generic flatness (Corollary 1.13.1), there is open dense subset U of G such that $\phi|_U$ is flat. Of course, any translate of U is as good as U. Assuming for a moment K is ACF, we may cover G by translates of U. This proves flatness of ϕ. If K is not ACF, we base change to $\overline{K}$, and since flat satisfies fppf descent (actually fpqc descent), see Stack Project, Tag 02YJ, we see ϕ is flat over K iff $\phi_{\overline{K}}$ is flat.

Next, assume $\dim(G) = \dim(H)$, then there is an open dense subset U' of H such that ϕ induces a fintie map $U := \phi^{-1}(U') \to U'$ whose fibers have cardinality equal the separable degree of $K(G)$ over $K(H)$. Also, this cardinality equals $|\ker(\phi)|$. Again, we assume K is ACF to cover G by translates of U proving finiteness of ϕ overall, and if K is not ACF, we can prove this by a base change as we have fppf descent.

A rational curve is a curve birational to $\mathbb{P}^1_K$. A variety is rationally connected if any two points in $X(\overline{K})$ may be connected by a rational curve over $\overline{K}$. It follows from Constancy Lemma 1.5 that abelian varieties do not contain rational curves. In particular, a morphism $X \to A$ into abelian variety contracts the rational curves of X to points. It follows that any morphism of a rationally connected variety, such as $\mathbb{P}^n$, into abelian variety is constant.

Proposition 1.15

Any morphism $f : \mathbb{P}^1_K \to G$ of the projective line into a group variety is constant.

Proof. Let $(x_0 : x_1)$ be homogeneous coordinates on $\mathbb{P}^1_K$. The map $\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{P}^1$ given by $((x_0 : x_1), y) \mapsto (x_0 : (x_1 + x_0 y))$ is a morphism. Now let $f : \mathbb{P}^1 \to G$ be a morphism, we apply Corollary 1.5.1 to the composition

$$\mathbb{P}^1 \times \mathbb{A}^1 \xrightarrow{s} \mathbb{P}^1 \xrightarrow{f} G$$

and see $f \circ s$ factors as $f(s(x, y)) = g(x)h(y)$ for two suitable morphisms $g : \mathbb{P}^1 \to G$ and $h : \mathbb{A}^1 \to G$.

We set $y = 0$ and note $s(x, 0) = x$, i.e. $g(x) = f(x)h(0)^{-1}$. Thus

$$f(s(x, y)) = f(x)h(0)^{-1}h(y)$$

Next set $x = \infty$, we see $s(\infty, y) = \infty$ and hence

$$f(\infty) = f(\infty)h(0)^{-1}h(y)$$

This shows $h(y) = h(0)$, i.e. h is a constant map and $f(s(x, y)) = f(x)$. Finally, take $x = 0$ we see $s(0, y) = y$ and so $f(y) = f(0)$.

Corollary 1.15.1

Let $U \subseteq \mathbb{P}^1_K$ be open and A be an abelian variety. Then $f : U \to A$ is constant for any f.

Proof. By valuative criterion of properness f extends to a morphism $\mathbb{P}^1 \to A$.

Q.E.D.

Theorem 1.16

Let $\phi : X \dashrightarrow G$ be rational map of smooth X into group variety G and $U_{\max}$ the domain of ϕ. Then every irreducible component of $X \backslash U_{\max}$ is of codimension 1.

Corollary 1.16.1

A ratinoal map from a smooth variety to an abelian variety is a morphism.

Proof. Let $\phi : X \dashrightarrow A$ be a rational map with domain $U_{\max}$. By valuative criterion of properness, $X \backslash U_{\max}$ has codimension at least 2. But then we see $U_{\max} = X$ by Theorem 1.16.

Q.E.D.

Our next goal is to prove the differential of multiplication on a group variety is given by addition.

Proposition 1.17

Let $m : G \times G \to G$ be multiplication of a smooth group variety G. Then the differential of m at ϵ is the map $T_{G,\epsilon} \oplus T_{G,\epsilon} \to T_{G,\epsilon}$ given by addition of tangent vectors.

Proof. In general we have $T_{X \times X',(x,x')} = T_{X,x} \oplus T_{X',x'}$. Thus $T_{G\times G,(\epsilon,\epsilon)} = T_{G,\epsilon} \oplus T_{G,\epsilon}$. For $\partial \in T_{G,\epsilon}$, we have

$$dm(\partial, 0) = dm \circ d\iota(\partial)$$

where $\iota : G \to G \times G$ is given by $g \mapsto (g, \epsilon)$. Since $dm \circ d\iota = d(m \circ \iota)$, we conclude $dm(\partial, 0) = \partial$. In the same say, we prove $dm(0, \partial) = \partial$. By linearity of dm, this gives the claim.

Q.E.D.

Corollary 1.17.1

Let G be smooth group variety and for $n \in \mathbb{Z}$, let $[n] : G \to G$ be the map $x \mapsto x^n$. Then the differential of $[n]$ at ϵ is the endomorphism of $T_{G,\epsilon}$ given by multiplying tangent vectors with n.

Proposition 1.18

Let G be an irreducible smooth group variety. Then the tangent bundle T_G on G is a trivial vector bundle of rank equal $\dim(G)$.

Proof. Let $\partial_\epsilon \in T_{G,\epsilon}$. By translation, we extend ∂_ϵ to a vector field ∂ on G. More precisely, let $\tau_x(y) := yx$ be right translation on G and $\partial_x(f) = \partial_\epsilon(f \circ \tau_x)$ for any $x \in G$ and $f \in \mathscr{O}_{G,x}$. Standard arguments for derivatives show ∂ is a vector field on G. Clearly, linearly independent tangent vectors in ϵ extends to vector fields, which are linearly independent in every fiber.

Q.E.D.