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1 Symmetric Functions

This is a course on asymmetric functions, with instructor Oliver Pechenik at University of Waterloo, in Fall 2021.

The universe of this course will be $\mathbb{Z}[x_1, ..., x_n]$. In particular, in symmetric function theory, we will be looking at

$$\mathbb{Z}[x_1, ..., x_n]^{S_n}$$

which is the ring of invariant of S_n, where we use $s_i := (i, i+1)$ as set of generators. In particular, we will consider three different actions of S_n on $\mathbb{Z}[X]$:

1. Permute the coordinate, i.e. $\pi \cdot f(x_1, ..., x_n) = f(x_{\pi(1)}, ..., x_{\pi(n)})$. In this case, the ring of invariant is symmetric polynomials $\mathrm{Sym}_n = \Lambda_n$.
2. Permute the coordinate only when one exponent is zero and one is not, i.e. the action of s_i on monomials is given by

$$s_i \cdot x_i^a x_{i+1}^b = \begin{cases} x_i^b x_{i+1}^a, & \text{if } a, b \text{ not both positive} \\ x_i^a x_{i+1}^b, & \text{otherwise} \end{cases}$$

   Then the ring of invariant is called ***quasi-symmetric polynomials***, and its denoted by QSym_n.
3. The trivial action, i.e. $\pi(f) = f$ and the ring of invariant is $\mathbb{Z}[X]$, we call this ***asymmetric polynomials*** and its denoted by ASym_n.

It turns out that all the things we can do in symmetric functions, have its counter-part in

$$\text{cohomology theory} \subseteq K\text{-theory} \subseteq \text{elliptic cohomology} \subseteq \text{cobordism}$$

In this semester, we will be dealing with cohomology (of Grassmannian), K-theory, with and without the adjective "equivariant".

Remark 1.1. Some notations:

1. λ, μ, γ for partitions.
2. α, β, γ for compositions.
3. a, b, c for weak compositions.
4. $X^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$.

To start with, we know Sym_n is $\mathbb{Z}$-module/$\mathbb{Q}$-vector space, but its infinite dimensional. Hence, we are going to consider its graded pieces, i.e.

$$\mathrm{Sym}_n = \bigoplus_{m \geq 0} \mathrm{Sym}_n^{(m)}$$

where $\mathrm{Sym}_n^{(m)}$ contains all symmetric polynomials with homogenous degree k.

Thus, what is the dimension of $\mathrm{Sym}_n^{(m)}$? Well, given any monomial of degree m, we know its not in $\mathrm{Sym}_n^{(m)}$, since it need to be symmetric. Thus the minimal symmetric polynomial contains the given monomial $\prod_{i=1}^{n} x_i^{u_i}$ would be, we apply all permutations to it and add it together, and for all repeating terms we only keep one of them in the sum.

This is the monomial symmetric polynomials m_λ, where λ is any partition. In particular, m_λ spans Sym_n and they are linearly independent, hence m_λ forms a basis of Sym_n. In particular, this implies

$$\dim_{\mathbb{Z}} \mathrm{Sym}_n^{(m)} = \text{number of partitions of } m \text{ with at most } n \text{ parts}$$

However, this is not the "good" basis. To see what would be a good basis, we consider a map of S_n on $\mathbb{Z}[X]$:

1. For $\pi \in S_n$, we have $\pi \cdot f(x_1, ..., x_n) = \mathrm{sgn}(\pi) f(x_{\pi(1)}, ..., x_{\pi(n)})$

The set (this is not a ring!) of polynomials satisfy the above condition are denoted by $V_n \mathrm{Sym}_n$, the alternating polynomials. In particular, inside this set, by setting any 2 variables equal gives us 0.

There are two particular alternating polynomials we can look at. The first one being the Vandermonde polynomial

$$v_n = \prod_{1 \le i < j \le n} (x_i - x_j)$$

This has the property that v_n divides all alternating polynomials, as if we set any 2 variables gives us 0 and hence it contains the factor $x_i - x_j$. In particular, we see for any alternating polynomial α we have

$$\alpha(X) = v_n(X) \cdot g(X)$$

where by apply permutations to the above equation we will see $g(X)$ must be symmetric.

In particular, if we consider the minimal alternating polynomial $\tilde{j}_a$ containing X^a, then we see if a has repeated value then $\tilde{j}_a = 0$. Thus, we might consider strict partitions, which are partitions without repeat and we get a basis for $V_n \mathrm{Sym}_n$, which are $\tilde{j}_\theta$ for θ run over all strict partitions.

Moreover, if we set $\delta = (n-1, n-2, ..., 0)$, then every strict partition looks like

$$\theta = \delta + \lambda$$

where λ are just any partitions. This is clearly a bijection. Hence we see

$$\dim(V_n \mathrm{Sym}_n^{(m + \binom{n}{2})}) = \dim \mathrm{Sym}_n^{(m)}$$

where the $\binom{n}{2}$ is given by Vandermonde polynomial v_n. Since the two vector spaces are isomorphic, we see by multiply/divide v_n, we get an explicit isomorphism between the two vector spaces.

By the isomorphism, we see the appropriate notation for $\tilde{j}_\theta$ should be j_λ, where $\theta = \delta + \lambda$.

Definition 1.2. The ***Schur polynomial*** s_λ is

$$s_\lambda := j_\lambda / v_n$$

Using Schur polynomials, we can get two more bases of Sym, which are the following definition.

Definition 1.3. The ***complete homogenous symmetric polynomials*** are given by $h_\lambda := \prod s_{\lambda_i}$.

Definition 1.4. The ***elementary symmetric polynomials*** are given by $e_\lambda := \prod s_{1^{\lambda_i}}$ where $1^{\lambda_i} = 111...1$ is the partition of λ_i many 1.

Theorem 1.5 (Newton)**.** *$\{e_\lambda\}, \{h_\lambda\}$ are bases of* Sym*.*

Definition 1.6. For bases A, B of a free $\mathbb{Z}$-module, say A ***refines*** B if every element of B is a linear combination elements of A with non-negative integer coefficients. We use $B \to A$ to denote this.

Theorem 1.7. *The* Sym *basis satisfy*

$$\begin{tikzcd}
\{e_\lambda\} \arrow[dr] & & \\
& \{s_\lambda\} \arrow[r] & \{m_\lambda\} \\
\{h_\lambda\} \arrow[ur] & &
\end{tikzcd}$$

Definition 1.8. A ***semistandard tableau*** T is a filling of the Young diagram of λ of positive integers such that the rows are weakly increasing from left to right and strictly increasing from top to bottom. This is denoted by SST or SSYT.

Theorem 1.9 (Littlewood)**.**

$$s_\lambda = \sum_{T \in \mathrm{SST}(\lambda)} x^{\mathrm{wt}(T)}$$

where $\mathrm{wt}(T) = (\#1s, \#2s, \ldots)$.

Definition 1.10. Let V be an $\mathbb{Z}$-algebra with $\mathbb{Z}$-basis $v_1, \ldots, v_n$, define ***structure coefficients*** c_{ij}^k by $v_i \cdot v_j = \sum_k c_{ij}^k v_k$.

We would love to have the c_{ij}^k to be non-negative integers. In our Sym case, we have $\{m_\lambda\}, \{e_\lambda\}$ and $\{h_\lambda\}$ to be non-negaitve integers. For e_λ and h_λ it is easy to see, e.g. $e_\lambda e_\mu = e_{\lambda \cup \mu}$ where $\lambda \cup \mu$ just means concatenating the two partitions and sort them.

Theorem 1.11 (A Littlewood-Richardson Rule). *Suppose $s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$. Then $c^\nu_{\lambda\mu}$ is equal the number of ballot semistandard tableaux of shape ν/λ and content μ.*

Well, there are a lot of new words in the above theorem. So here goes definitions.

Definition 1.12. Let ν, λ be two partitions, then ν/λ is the Young diagram of ν set minus the Young diagram of λ.

Definition 1.13. A ***content*** of a filling for a Young diagram is given by $(a_1, a_2, \ldots)$ where a_i means a_i many i's in the filling.

Next we need to define what ballot means.

Definition 1.14. For a tableau we define the ***reading word*** as read the tableau from top to bottom, right to left. E.g.

$$\begin{ytableau} 1 & 1 & 2 \\ 2 & 4 \end{ytableau} \Rightarrow 21142$$

Then we call the reading word ***ballot***, if when we read two leading words left to right, we always have seen at least as many i's as $(i+1)$'s for all i.

Example 1.15. We compute $c^{321}_{21,21}$. Thus the skew diagram looks like

$$\begin{ytableau} \none & \none & \\ \none & \\ \\ \end{ytableau}$$

This correspond to the following three tableaux

$$\begin{ytableau} \none & \none & 1 \\ \none & 1 \\ 2 \end{ytableau} \qquad \begin{ytableau} \none & \none & 1 \\ \none & 2 \\ 1 \end{ytableau} \qquad \begin{ytableau} \none & \none & 2 \\ \none & 1 \\ 1 \end{ytableau}$$

Next we need to check ballot condition, and this rule out the last one 211. Hence $c^{321}_{21,21} = 2$.

Next, we talk about little bit of K-theory.

Definition 1.16. We use $\mathrm{SV}(\lambda)$ to denote the set of ***set-valued semistandard tableaux***, which is a filling of the Young diagram of λ by non-empty sets of positive integers, such that However we delete all but one entry of each box, we get a semistandard tableau, e.g. if we have

$$\begin{ytableau} A & B \\ C \end{ytableau}$$

then we must have $\max(A) \le \min(B)$ and $\max(A) < \min(C)$.

Definition 1.17. The ***symmetric β-Grothendieck polynomial*** is

$$g_\lambda = \overline{s_\lambda} = \sum_{T \in \mathrm{SV}(\lambda)} \beta^{\mathrm{ex}(T)} x^{\mathrm{wt}(T)}$$

where ex is another weight function given by $\mathrm{ex}(T)$ equal the number of extra numbers.

Example 1.18. We consider $\overline{s_{21}}(x_1, x_2)$. Then we have three set-valued semistandard tableaux

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1,2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

so

$$\overline{s_{21}} = x_1^2 x_2 + x_1 x_2^2 + \beta x_1^2 x_2^2$$

where we note if we set $\beta = 0$ we get s_{21} back.

Similarly we have five elements in $\mathrm{SV}(31)$, given by

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1,2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1,2 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$$

A quick way to check whether we get all the $\mathrm{SV}(\lambda)$, we can compute $\overline{s_\lambda}(\beta = -1, x_i = 1)$, then normally the value should be 1.

Theorem 1.19 (Pechenik-Yong)**.** *We have*

$$\overline{s_\lambda} \cdot \overline{s_\mu} = \sum_\nu C^\nu_{\lambda,\mu} \overline{s_\nu}$$

where $C^\nu_{\lambda,\mu}$ is $\beta^{|\nu|-|\mu|-|\lambda|}$ multiply the number of ballot genomic tableaux of shape ν/λ and content μ.

This time, ν/λ is still the skew diagram, but we are going to get $|\nu| \ge |\lambda| + |\mu|$ as we are taking set values. Next, we need to explain what genomic means.

Definition 1.20. A ***gene*** of a family i in a semistandard tableau is a collection of labels i that are consecutive in left-to-right order with no two in the same row.

Definition 1.21. A ***genomic*** tableau is a semistandard tableau with a partition into genes.

The ***content*** in this case is given by $(\#\text{ of 1 genes}, \#\text{ of 1 genes}, \ldots)$. For example,

		1	2
1	1	2	
2			

would have content $(2,1)$ as we have two 1 genes and one 2 gene.

A ***genotype*** is a choice of one box per gene. Then a genomic tableau is ***ballot*** if all the genotypes are.

2 Quasisymmetric Functions

Definition 2.1. Let $f \in \mathbb{Z}[x_1, ..., x_n]$, then f is ***quasisymmetric*** if for all $(a_1, ..., a_k), (i_1 < ... < i_k), (j_1 < ... < j_k) \in \mathbb{Z}^k$, the coefficient of $x_{i_1}^{a_1} ... x_{i_k}^{a_k}$ is equal the coefficient of $x_{j_1}^{a_1} ... x_{j_k}^{a_k}$.

To start with, we can do something similar to monomial symmetric polynomials. For example, if we start with $x_1 x_3^2 \in \mathbb{Z}[x_1, x_2, x_3]$, then we see we must add $x_1 x_2^2$ and $x_2 x_3^2$ in it to make the polynomial quasisymmetric. In this case, the best index should be $x_1 x_2^2$.

Definition 2.2.

1. A ***weak composition*** is a finite string of nonnegative integers.
2. A ***(strong) composition*** is a finite string of positive integers.
3. For a weak composition b we define the ***positive part of*** b, denoted by b^+, to be the composition obtained by delete all zero in b.

Definition 2.3. For every strong composition α, we define the ***monomial quasisymmetric polynomial*** $M_\alpha(x_1, ..., x_n)$ to be

$$M_\alpha(x_1, ..., x_n) = \sum_b x^b$$

where b range over weak compositions of length n with $b^+ = \alpha$.

One should convince oneself M_α form a basis. Also, we should expect the structure coefficients to be positive.

Let α be composition with length m and β be with length n. Then $\alpha\beta$ is the composition obtained by concatenation. Next, consider a value k with $\max(m,n) \le k \le m+n$ and a surjection $t : [m+n] \to [k]$ such that $t(i) < t(j)$ for $i < j \le m$ or $m < i < j$.

For the choice of k and $t : [m+n] \to [k]$, we get the corresponding ***overlapping shuffle*** of α, β is $\gamma(t)$ given by $\gamma_i = \sum_{t(j)=i} (\alpha\beta)_j$. We write $\alpha \sqcup\!\sqcup_o \beta$ for the formal sum $\sum_{k,t} \gamma(t)$.

Example 2.4. We have $(2)\,ш_0(1,2) = (2,1,2) + 2(1,2,2) + (3,2) + (1,4)$.

Theorem 2.5 (Hazewinkel)**.**

$$M_\alpha \cdot M_\beta = \sum_\gamma c^\gamma_{\alpha\beta} M_\gamma$$

where $c^\gamma_{\alpha\beta}$ is multiplicity of γ in $\alpha\,ш_0\,\beta$.

Example 2.6. For example, we should have $M_2M_{12} = M_{212} + 2M_{122} + M_{32} + M_{14}$. One can indeed verify by direct computation.

Definition 2.7. Say β ***refines*** α if α can be obtained from β by summing consecutive entries and we write $\beta \vDash \alpha$.

Example 2.8. We see $(1,2,1) \vDash (1,3)$ and $(1,2,1) \vDash (3,1)$ but $(2,1,1) \nvDash (1,3)$.

Definition 2.9. The ***fundamental quasisymmetric polynomials***

$$F_\alpha(x_1, ..., x_n) = \sum_b x^b$$

ranging over weak compositions of length n with $b^+ \vDash \alpha$.

Example 2.10. We see

$$F_{13}(x_1, x_2, x_3) = x^{130} + x^{103} + x^{013} + x^{112} + x^{121} = M_{13} + M_{112} + M_{121}$$

Theorem 2.11. *The M_α basis refines the F_α basis, i.e. $\{F_\alpha\} \to \{M_\alpha\}$. In particular, $F_\alpha = \sum_{\beta \vDash \alpha} M_\beta$.*

The remarkable thing is F_α also have positive structure coefficients.

Definition 2.12. Let A, B be two words on disjoint alphabets $\mathcal{A}, \mathcal{B}$, then a ***shuffle*** of A, B is a permutation of AB such that the subword on $\mathcal{A}$ is A and on $\mathcal{B}$ is B.

In other word, if $|A| = m, |B| = n$, then shuffle is bijection $s : [m+n] \to [m+n]$ such that $s(i) < s(j)$ when $i < j \le m$ or $m \le i < j$.

Now let α, β be two compositions.

Set $\mathcal{A}$ be the alphabet of odd integers and $\mathcal{B}$ the even integers.

Then we define A to be α_1 copies of $2\ell(\alpha) - 1$, α_2 copies of $2\ell(\alpha) - 3$,...,$\alpha_{\ell(\alpha)}$ copies of 1. For example, if $\alpha = (2,1,3,4)$ then A should be 7753331111.

Similarly we define B to be β_1 copies of $2\ell(\beta)$, β_2 copies of $2\ell(\beta) - 2$,..., $\beta_{\ell(\beta)}$ copies of 2. For example, if $\beta = (2,1,3,4)$ then B is 8864442222.

Then let $A\,ш\,B$ be the set of shuffles of A and B. Note $|A\,ш\,B| = \binom{|\alpha|+|\beta|}{|\alpha|}$.

For $C \in A\,ш\,B$, we break C into maximal runs of weakly increasing entries. The ***descent composition*** $\mathrm{des}(C)$ has $\mathrm{des}(C)_i$ equal the length of the ith run.

Definition 2.13. The ***shuffle product*** (due to Eilenberg and MacLane) is the formal sum $\alpha \shuffle \beta = \sum_{C \in A \shuffle B} \mathrm{des}(C)$.

Example 2.14. Let $\alpha = (2), \beta = (1,2)$, then $A = 11$ and $B = 422$. Then we have

$$\begin{aligned} &A \shuffle B \\ &= 4|22|11 + 4|2|12|1 + 4|122|1 + 14|22|1 \\ &\quad + 4|2|112 + 4|12|12 + 14|2|12 + 4|1122 + 14|122 + 114|22 \end{aligned}$$

and so

$$\alpha \shuffle \beta = 122 + 1121 + 131 + 221 + 113 + 122 + 212 + 14 + 23 + 32$$

Theorem 2.15.
$$F_\alpha \cdot F_\beta = \sum_\gamma C^\gamma_{\alpha\beta} F_\gamma$$
where $c^\gamma_{\alpha\beta}$ is the multiplicity of γ into the shuffle product $\alpha \shuffle \beta$.

Example 2.16. If we have four variables, then $F_2 F_{12} = 2F_{122} + F_{1121} + F_{131} + F_{221} + F_{113} + F_{212} + F_{14} + F_{23} + F_{32}$.

Now recall the symmetric group has generators $s_i = (i, i+1)$ with $1 \le i \le n-1$ with relations:

1. $s_i^2 = \mathrm{Id}$
2. $s_i s_j = s_j s_i$ if $|i - j| \ge 2$
3. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

This is all the relations of S_n, and from this we get a similar definition. Then the representations of S_n correspond to symmetric polynomials (this map is called the Frobenius map), and irreducible representations correspond to Schur polynomials.

Definition 2.17. The ***0-Hecke algebra*** has generators τ_i for $1 \le i \le n-1$ with relations:

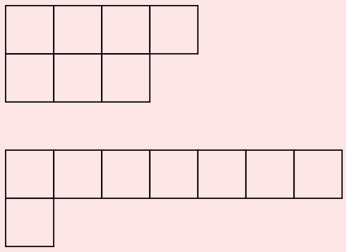
1. $\tau_i \tau_j = \tau_j \tau_i$ if $|i - j| \ge 2$
2. $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$
3. $\tau_i^2 = \tau_i$

In the case of 0-Hecke algebra, representations of 0-Hecke algebra correspond to quasisymmetric polynomials, and irreducible representations (they are all 1-dimensional) correspond to F_α. However, in this case we can get indecomposable representations of dimension greater than 1.

Last time we learned M_α and F_α. Today we will talk about ***quasisymmetric Schur polynomials*** (or ***quasiSchur polynomial***) due to Haglund, Luoto, Mason, Van Willigenburg in 2011.

The combinatorial structure of this will be weird, but they actually come from specialization of MacDonald polynomials.

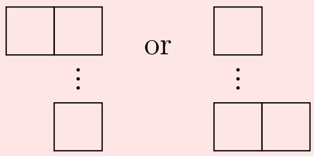
First, if we have a weak composition, we can still make a Young diagram associated with it, e.g. if we have $(4, 3, 0, 7, 1)$ then the diagram is



In some convention, we will also add a "basement" at column 0.

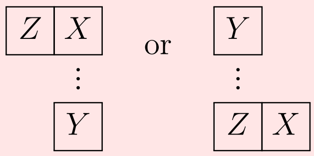
Definition 2.18. A ***composition tableau*** of shape a is an assignment of positive integers to the cells of the Young diagram of weak composition a.

Definition 2.19. A ***triple*** of boxes is a set of 3 boxes in the configuration



where at left the upper row weakly longer and at the right the upper row strictly shorter.

A triple is ***inversion*** if, when we label the above triple by the following



we ***don't*** have $X \leq Y \leq Z$.

Definition 2.20. A composition tableau is ***semistandard*** if:

1. entries don't repeat in a column.
2. rows weakly decrease left to right.
3. every triple is inversion.
4. entries in the first column (not the basement) equal the row indices.

Definition 2.21. For a weak composition a, write $\mathfrak{A}\mathrm{SST}(a)$ for the set of semistandard tableau of shape a. Then we define the ***Demazure atom***

$$\mathfrak{A}_a = \sum_{T \in \mathfrak{A}\mathrm{SST}(a)} x^{\mathrm{wt}(a)}$$

Definition 2.22. The ***quasiSchur*** polynomial

$$S_\alpha(x_1,\dots,x_n) = \sum_{a^+=\alpha} \mathfrak{A}_a = \sum_{a^+=\alpha} \sum_{T\in\mathfrak{A}\mathrm{SST}(a)} x^{\mathrm{wt}(T)}$$

where the latter sum is over weak composition of length n.

Example 2.23.

$$S_{13}(x_1,x_2,x_3) = \mathfrak{A}_{013} + \mathfrak{A}_{103} + \mathfrak{A}_{130}$$

Next we need to find each Demazure atoms.

We start with 130, which have the shape

$$\begin{array}{|c|c|c|} \cline{1-1} \\ \hline & & \\ \hline \end{array}$$

By rule 4 of semistandard composition tableau, we see we must have

$$\begin{array}{|c|c|c|} \cline{1-1} 1 \\ \hline 2 & & \\ \hline \end{array}$$

But by rule 2 we see we already don't have much choice. Next, we see we only have one triple, which is the one contains 1 and 2. Since we don't want $X \leq Y \leq Z$, hence we must have

$$\begin{array}{|c|c|c|} \cline{1-1} 1 \\ \hline 2 & 2 & \\ \hline \end{array}$$

and the last box can be both 1 or 2. Thus we end up with the following

$$\begin{array}{|c|c|c|} \cline{1-1} 1 \\ \hline 2 & 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \cline{1-1} 1 \\ \hline 2 & 2 & 1 \\ \hline \end{array}$$

Next, we consider 103. The shape is

$$\begin{array}{|c|c|c|} \cline{1-1} 1 \\ \cline{1-1} \multicolumn{3}{c}{} \\ \hline 3 & & \\ \hline \end{array}$$

With similar argument, we see we have five possibilities here:

$$\begin{array}{|c|c|c|} \cline{1-1} 1 \\ \cline{1-1} \multicolumn{3}{c}{} \\ \hline 3 & 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \cline{1-1} 1 \\ \cline{1-1} \multicolumn{3}{c}{} \\ \hline 3 & 3 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \cline{1-1} 1 \\ \cline{1-1} \multicolumn{3}{c}{} \\ \hline 3 & 3 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \cline{1-1} 1 \\ \cline{1-1} \multicolumn{3}{c}{} \\ \hline 3 & 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \cline{1-1} 1 \\ \cline{1-1} \multicolumn{3}{c}{} \\ \hline 3 & 2 & 1 \\ \hline \end{array}$$

For 013, it is also similar and we get

$$\begin{array}{|c|cc} \hline 2 & \multicolumn{2}{c}{} \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 2 & \multicolumn{2}{c}{} \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 2 & \multicolumn{2}{c}{} \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

Thus, we see their corresponding generating function would be

$$\begin{aligned} S_{13}(x_1, x_2, x_3) = (&x_1 x_2^3 + x_1^2 x_2^2) \\ &+ (x_1 x_3^3 + x_1 x_2 x_3^2 + x_1^2 x_3^2 + x_1 x_2^2 x_3 + x_1^2 x_2 x_3) \\ &+ (x_2 x_3^3 + x_2^2 x_3^2 + x_1 x_2 x_3^2) \end{aligned}$$

which is indeed quasisymmetric.

Theorem 2.24. $S_\alpha \in \mathrm{QSym}$

Proof. We want to show S_α is invariant under $x_i \leftrightarrow x_{i+1}$ in monomial that without both x_i and x_{i+1}.

Let $X_{\alpha,i}$ is equal the semistandard tableaux of shape a with $a^+ = \alpha$ and i not appearing.

Define $\phi : X_{\alpha,i+1} \to X_{\alpha,i}$ by replacing all i's with $(i+1)$'s and moving row i into row $i+1$.

Thus we want to check the map is well-defined and land in $X_{\alpha,i}$, and the inverse map. Thus we look at the four conditions in definition of semistandard composition tableaux:

1. (1) is clearly satisfied.
2. (2) is fine as well, as we see if we have (a, i, b) in a row, then since we don't have $i+1$ we have $a > i+1$ and so $a \geq i+1$ and we don't care about b. Hence (2) is satisfied.
3. (3) is fine if you think about it as it only depend on relative order.
4. (4) is clearly satisfied.

By the same reasoning, we see the inverse map is also well-defined, thus we are done the proof. ♡

Proposition 2.25. *The m_λ expand positively in the M_α basis in* QSym*, i.e.*

$$m_\lambda(x_1, \dots, x_n) = \sum_{\mathrm{sort}(\alpha)=\lambda} M_\alpha(x_1, \dots, x_n)$$

Theorem 2.26. *The s_λ expand positively in the S_α basis. In particular we have* $s_\lambda(x_1, \dots, x_n) = \sum_{\mathrm{sort}(\alpha)=\lambda} S_\alpha(x_1, \dots, x_n)$

Theorem 2.27. *We have*

$$\begin{tikzcd} \{S_\alpha\} \arrow[r] & \{F_\alpha\} \arrow[r] & \{M_\alpha\} \\ \{s_\lambda\} \arrow[u] \arrow[rr] & & \{m_\lambda\} \arrow[u] \end{tikzcd}$$

Example 2.28. We continue the example and try to expand s_{31} using S_α. Well, we already know $S_{13}(x_1, x_2, x_3)$, and next we compute $S_{31} = \mathfrak{A}_{310} + \mathfrak{A}_{301} + \mathfrak{A}_{031}$. For 310, we get

$$\begin{ytableau} 1 & 1 & 1 \\ 2 \end{ytableau}$$

and for 301 we get

$$\begin{ytableau} 1 & 1 & 1 \\ \none \\ 3 \end{ytableau}$$

and 031 we have three:

$$\begin{ytableau} 2 & 2 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 2 & 2 & 1 \\ 3 \end{ytableau} \quad \begin{ytableau} 2 & 1 & 1 \\ 3 \end{ytableau}$$

Sum together we get

$$S_{31}(x_1, x_2, x_3) = (x_1^2 x_2) + (x_1^3 x_3) + (x_2^3 x_3 + x_1 x_2^2 x_3 + x_1^2 x_2 x_3)$$

Now we add $S_{13}(x_1, x_2, x_3)$ and $S_{31}(x_1, x_2, x_3)$.

Theorem 2.29. $\{S_\alpha(x_1, ..., x_n)\}$ *is a basis of* QSym_n.

Proof. Take the reverse lexicographic order on monomial, i.e. $x^a < x^b$ iff $b - a$ has rightmost non-zero entry is positive.

Notice $\mathfrak{A}_a = x^a +$ smaller stuff, so

$$S_\alpha = x^{0^{n-\ell(\alpha)}\alpha} + \text{smaller stuff}$$

where $0^{n-\ell(\alpha)}\alpha$ is concatenating a bunch of 0 in front then put α at the end. This immediately tell us S_α is linearly independent (since each α would have a unique $0^{n-\ell(\alpha)}\alpha$) and spanning (since each monomial quasisymmetric must contain one of $x^{0^{n-\ell(\alpha)}\alpha}$). ♡

Last time we were talking about quasiSchur polynomials, and we proved they are quasisymmetric and form a basis. In the refinment diagram

$$\begin{tikzcd}
\{S_\alpha\} \arrow[r] & \{F_\alpha\} \arrow[r] & \{M_\alpha\} \\
\{s_\lambda\} \arrow[u] \arrow[rr] & & \{m_\lambda\} \arrow[u]
\end{tikzcd}$$

we know four of the five arrows, and we are only missing $\{S_\alpha\} \to \{F_\alpha\}$. However, we also want to know the arrows between $\{s_\lambda\}$ and $\{F_\alpha\}$, i.e. we want to know the dash arrow in the following:

$$\begin{tikzcd}
\{S_\alpha\} \arrow[r] & \{F_\alpha\} \arrow[r] \arrow[dl, dotted, shift left] & \{M_\alpha\} \\
\{s_\lambda\} \arrow[u] \arrow[rr] \arrow[ur] & & \{m_\lambda\} \arrow[u]
\end{tikzcd}$$

Well, before this, we study $\{S_\alpha\} \to \{F_\alpha\}$.

Definition 2.30. We say T is ***initial*** if the integers i appearing in T are an initial segment of $\mathbb{Z}^+$.

Example 2.31. We reuse the example of S_{13}. We see the following are initial:

$$\begin{array}{|c|cc} \hline 1 & \multicolumn{2}{|c}{} \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 1 & \multicolumn{2}{|c}{} \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 1 & \multicolumn{2}{|c}{} \\ \hline \multicolumn{3}{c}{} \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 1 & \multicolumn{2}{|c}{} \\ \hline \multicolumn{3}{c}{} \\ \hline 3 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 1 & \multicolumn{2}{|c}{} \\ \hline 3 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 2 & \multicolumn{2}{|c}{} \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

Definition 2.32. We say T is ***quasi-Yamanouchi*** (qY) if, for all $i \in T$, either i appears in column 1 or there is an $i+1$ weakly right[1] of an i.

Example 2.33. The following are qY:

$$\begin{array}{|c|cc} \hline 1 & \multicolumn{2}{|c}{} \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 1 & \multicolumn{2}{|c}{} \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 1 & \multicolumn{2}{|c}{} \\ \hline \multicolumn{3}{c}{} \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 1 & \multicolumn{2}{|c}{} \\ \hline \multicolumn{3}{c}{} \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|cc} \multicolumn{3}{c}{} \\ \hline 2 & \multicolumn{2}{|c}{} \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

Theorem 2.34.

$$S_\alpha = \sum_T F_{\mathrm{wt}(T)}$$

where the T is summing over all $T \in \bigcup_{a^+=\alpha} \mathfrak{A}\mathrm{SST}(a)$ such that T is initial and quasi-Yamanouchi.

[1] Not in the same row, but just on the right in general

Since we know F_α correspond to 1-dimensional irreducible representation of the 0-Hecke algebra, and S_α expand positively in F_α, each S_α is quasisymmetric Frobenius characters of some 0-Hecke module.

In particular, if

$$S_\alpha = \sum_\beta c_\alpha^\beta F_\beta$$

then S_α corresponds to $\bigoplus_\beta c_\alpha^\beta I_\beta$, where I_β's are irreducible representations.

However, this is boring, and we can do better. By Tewar and van Willigenburg (2015), we get a 0-Hecke module structure on the formal span of the $\mathfrak{A}$SST, but this is still not indecomposable. Thus, it is still an open problem to find an indecomposable module for S_α.

There is another sad thing. The S_α basis does not have positive structure constant. But, we have a weaker thing is true, that is, $s_\lambda \cdot S_\alpha$ is positive in quasiSchur (due to Haglund, Luoto, Mason, van Willigenburg). We will later find that, this seems always be the case, i.e. when we have a basis, even it may not have positive structure coefficient, but if you multiply with s_λ, its always positive.

Finally, we note the QSym bases have K-analogs:

1. The ***multi-monomial***: $\overline{M_\alpha} = M_\alpha +$ higher terms.
2. The ***multi-fundamental***: $\overline{F_\alpha} = F_\alpha +$ higher terms.
3. The ***quasiGrothendieck***: $\overline{S_\alpha} = S_\alpha +$ higher terms.

The first two are due to Lam and Pylyavskyy in 2007 and the third one is due to Monical in 2016.

3 Asymmetric Functions

This is going to be the end of quasisymmetric polynomials, and we are going to move on to asymmetric.

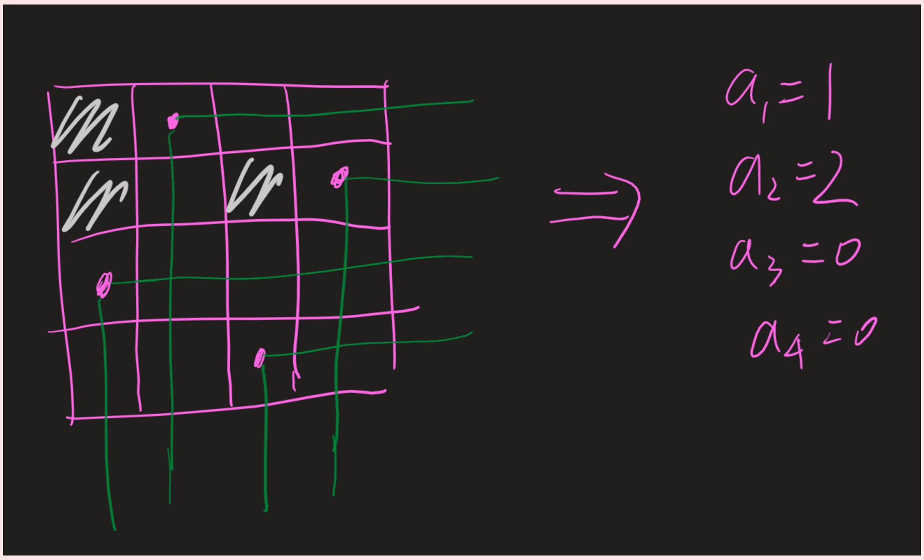
Now lets talk about ASym = $\mathbb{Z}[x_1, \dots, x_n]$. The most trivial basis of ASym is of course just given by all the oridinary monomials $x^a = \mathfrak{X}_a$ with a range over all weak composition. Thus $\dim \mathrm{ASym}_n^{(m)}$ (n variables and m degree homogenous) is equal the number of weak compositions of m with length n.

But it more traditional to label some of our bases by permutations. Here is how we going to translate between weak compositions and permutations.

Definition 3.1. For $\pi \in S_n$, let $a_i = \#\{j : j > i, \pi(j) < \pi(i)\}$, then we define $\mathrm{invcode}(\pi) = (a_1, \dots, a_n)$, which is called the ***Lehmer code***.

We note $a_i \leq n - i$ and $a_n = 0$. If we sum over the Lehmer code, we get the number of inversions, i.e. $\sum a_i = \mathrm{inv}(\pi)$, which is also called the coxeter length.

Example 3.2. If $\pi = 2413$, then $a_1 = 1, a_2 = 2, a_3 = 0, a_4 = 0$. There is a visual way to compute the Lehmer code as follows:



To get this, we first draw n by n grid, then at 1th row we place a laser gun at the $\pi(1)$th column, and so on. Then laser gun destory all the things at the right and down. Then we count whats left. The above white boxes diagram (not the entire grid!) is called a ***Rothe diagram*** of π and is denoted by $D(\pi)$.

We note:

1. $D(\pi^{-1})$ is equal the transpose of $D(\pi)$ as the lasers are symmetric.
2. $D(\pi)$ has ***northwest*** property, that is, if we have

C	$\dots$	B
$\vdots$		
A		

and $A, B \in D(\pi)$ then $C \in D(\pi)$.

Definition 3.3. The ***rank function*** $r(i,j)$ of a cell $(i,j) \in [n] \times [n]$ is equal the number of "lasers"/"dots" weakly northwest of (i,j). We note $r(i,j) = \#\{k \le i : \pi(k) \le j\}$.

Example 3.4. If $\pi = 2413$ then the rank functions are given by

0	1	1	1
0	1	1	2
1	2	2	3
1	2	3	4

Definition 3.5 (Fulton). The ***essential set*** of $D(\pi)$ is the maximally south-east boxes of each connected component of $D(\pi)$, i.e. $\operatorname{Ess}(\pi) = \{(i,j) \in D(\pi) : (i+1,j) \notin D(\pi), (i,j+1) \notin D(\pi)\}$.

Proposition 3.6. *$\pi \in S_n$ is determined by its Lehmer code, its Rothe diagram, or the restriction of its rank function to* $\operatorname{Ess}(\pi)$.

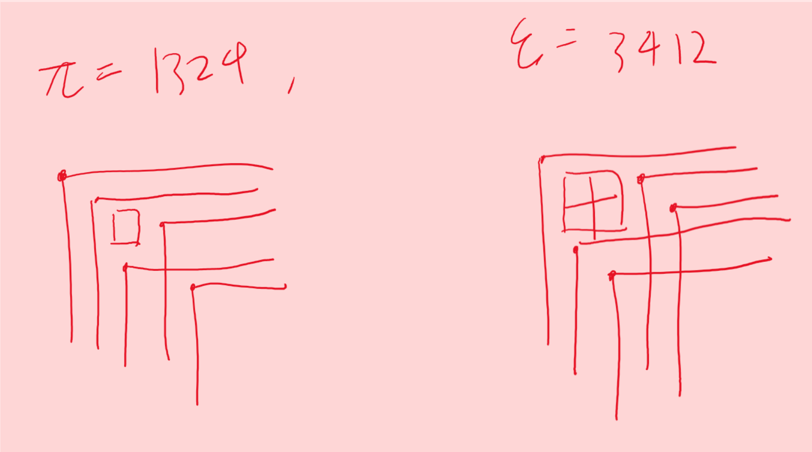
Proof. From $\operatorname{invcode}(\pi) = (a_1, a_2, \ldots, a_n)$, we recursively reconstruct π. We see we must have $\pi(1) = a_1 + 1$. Suppose we have determined $\pi(1), \ldots, \pi(k)$, then $\pi(k+1)$ is the $(a_{k+1}+1)$th smallest element of $[n] \backslash \{\pi(1), \ldots, \pi(k)\}$. Similarly if we know its Rothe diagram, then we get its Lehmer code and we are done.

Now suppose we only know $\operatorname{Ess}(\pi)$ and the rank function on it.

If there is no $b \in \operatorname{Ess}(\pi)$ with $r(b) = 0$, then $\pi(1) = 1$. Otherwise, let $(i,j) \in \operatorname{Ess}(\pi)$ be the rightmost with $r(i,j) = 0$. Then $\pi(1) = j+1$. Now suppose we $\pi(1), \ldots, \pi(k)$ are known. Look at boxes $(i,j) \in \operatorname{Ess}(\pi)$ with $i \geq k+1$ such that $r(i,j) = r(k,j)$ where $r(k,j)$ is determined by $\pi(1), \ldots, \pi(k)$ which we already know. Then $\pi(k+1)$ must be the smallest unused value strictly right of these boxes. ♡

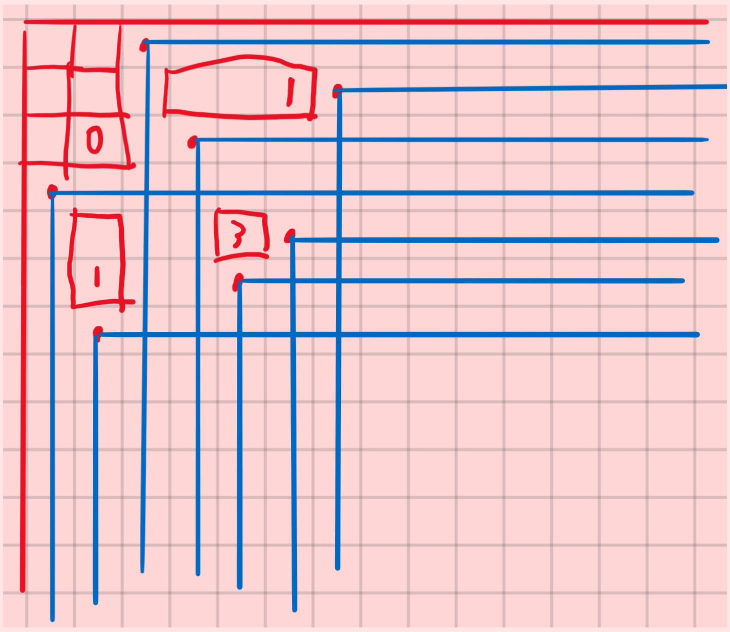
Last time we showed we only need some partial data to construct the permutation back. Today we start with some examples.

Example 3.7. First, we show that with only the essential set and not the rank function, it is not enough. The following two diagrams



both have the same essential set, but we could not tell they are different if we do not have the rank function.

Example 3.8. Second, we consider a large enough example to illustrate the construction using essential set:



One should try to work this out and see we indeed only have one way to do this.

Now recall S_n is generated by $s_1, \ldots, s_{n-1}$ where s_i swap i and $i+1$ with relations:

1. $s_i^2 = 1$
2. $s_i s_j = s_j s_i$ for $|i - j| \geq 2$
3. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

In this course, we set conventions that, if we have a sequence of integers, then act from the right we swap the position, and act from the left we swap the values. For example, we have

$$3741652 \cdot s_4 = 3746152 \quad s_4 \cdot 3741652 = 3751642$$

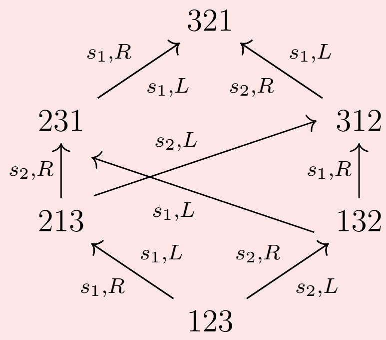
Suppose I write $\omega \in S_n$ as a product of s_i, how many factors do I need? Well, the minimal value we need is $\mathrm{inv}(\omega)$.

Definition 3.9. If $\omega = s_{i_1} s_{i_2} \ldots s_{i_k}$ with $k = \mathrm{inv}(\omega)$, then we call this a ***reduced expression*** and $i_1 i_2 \ldots i_k$ is a ***reduced word***. We write $R(\omega)$ equal the set of all reduced words.

Next, we define some partial orders on S_n:

1. Right weak order: we say $\mu \lessdot_R us_i$ if $\mathrm{inv}(us_i) = \mathrm{inv}(u) + 1$
2. Left weak order: we say $\mu \lessdot_R s_i u$ if $\mathrm{inv}(s_i u) = \mathrm{inv}(u) + 1$.
3. 2-sided weak order: union of left and right weak order.
4. Strong/Bruhat order: Let $t_{i,j} \in S_n$ be swap i and j, then $u \lessdot ut_{i,j}$ if $\mathrm{inv}(ut_{i,j}) = \mathrm{inv}(u) + 1$.

Example 3.10. The following is the right and left weak orders, where L and R indicates we apply the action to left or right.



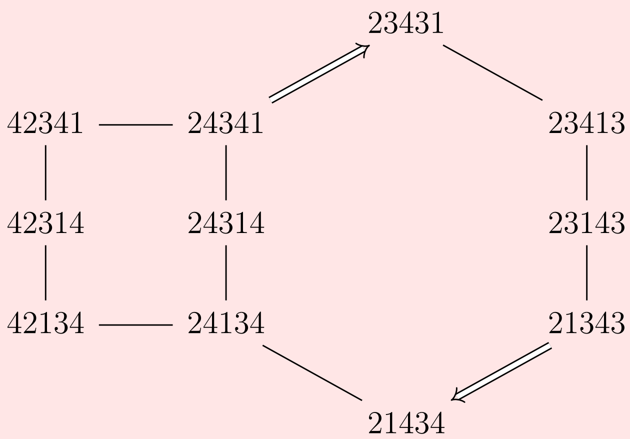
We also want to build a graph on the reduced words of ω with edges corresponding to commutation (i.e. $s_i s_j = s_j s_i$) and braids (i.e. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$).

Example 3.11. Say we have $\omega = 31542 \in S_5$, then one reduced word is 42341. The complete list is given by

$$[2,3,4,3,1], [4,2,3,4,1], [2,4,3,4,1], [2,3,4,1,3], [2,3,1,4,3]$$

$$[2,1,3,4,3], [4,2,3,1,4], [2,4,3,1,4], [4,2,1,3,4], [2,4,1,3,4], [2,1,4,3,4]$$

Now start with this word 42341, we use try to draw the graph as (one edge means commutation and double edge means braid):



We see this is actually all the reduced words. We define this graph as $\mathcal{G}(\omega)$.

Remark 3.12. It is a open problem that to give a short elementary proof to the fact $\mathcal{G}(\omega)$ is bipartite.

Theorem 3.13. *For all $\omega \in S_n$, the graph $\mathcal{G}(\omega)$ is connected.*

Proof. We will prove this after some lemmas. ♡

Lemma 3.14. *Suppose $i_1, i_2, ..., i_\ell \in R(\omega)$. Then the inversion of ω are*

$$I(\omega) = \{s_{i_\ell} ... s_{i_{h+1}}(i_h, i_h + 1) : 1 \le h \le \ell\}$$

Proof. First we consider an example. Say $\omega = 31542$, and we have $2343 \in R(\omega)$. Then say $h = 1$, and we have $s_1 s_3 s_4 s_3(2, 3) = (1, 5)$, which is indeed an inversion if we look at the 1th entry in ω and the 5th entry in ω. Next, say $h = 2$, then we get $s_1 s_3 s_4(3, 4) = (4, 5)$, which is indeed an inversion. If $h = 3$ then we have $s_1 s_3(4, 5) = (3, 5)$ and $h = 4$ we have $s_1(3, 4) = (3, 4)$. Last, we see $h = 5$ then $(1, 2)$ is indeed an inversion.

Next, we can do this all over again with another reduced word $42134 \in R(\omega)$. This time, we get

$$h = 1 \Rightarrow s_4 s_2 s_1 s_3(4, 5) = (3, 5)$$
$$h = 2 \Rightarrow s_4 s_2 s_1(2, 3) = (1, 5)$$
$$h = 3 \Rightarrow s_4 s_2(1, 2) = (1, 2)$$
$$h = 4 \Rightarrow s_4(3, 4) = (3, 5)$$
$$h = 5 \Rightarrow (4, 5)$$

Now we consider the actual proof. We use induction on the coxeter length. Clearly this holds for the identity (with coxeter length 0). Suppose $\mathrm{inv}(us_m) = \mathrm{inv}(u) + 1$, then $I(us_m) = \{(m, m+1)\} \cup s_m I_u$, which the induction follows. ♡

Lemma 3.15 (Exchange Lemma)**.** *Suppose $i_1 ... i_\ell, j_1 ... j_\ell \in R(w)$. Then there exists k such that $j_1 i_1 ... \hat{i_k} ... i_\ell$ is a reduced word.*

Proof. By the above lemma for w^{-1} with $j_1 ... j_\ell$, we see $(j_1, j_1 + 1)$ is an inversion for ω^{-1}, and there is some k such that $(j_1, j_1 + 1) = s_{i_1} ... s_{i_{k-1}}(i_k, i_k + 1)$ if we use the above lemma for w^{-1} with $i_1 ... i_\ell$. So $s_{i+1} ... s_{i_{k-1}} = s_{i_1} ... s_{i_{k-1}}$ and hence by the above equality about $(j_1, j_1 + 1) = s_{i_1} ... s_{i_{k-1}}(i_k, i_k + 1)$, we see we must have

$$s_{j_1} s_{i_1} ... s_{i_{k-1}} = s_{i_1} ... s_{i_{k-1}} s_k$$

At this point we are done, if we just append $s_{i_{k+1}} ... s_{i_\ell}$ at the end. ♡

Theorem 3.16. *For all $w \in S_n$, the graph $\mathcal{G}(w)$ of reduced words is connected.*

Proof. By induction on coxeter length of w. If the coxeter length is 0 or 1 we are done. Let $i_1 ... i_\ell, j_1 ... j_\ell$ be two reduced words. By Exchange Lemma, we see $j_1 i_1 ... \hat{i_k} ... i_\ell$ is a reduced word.

Case 1: Say $k \neq \ell$, then by induction, if we forget i_ℓ in both the words $i_1 ... i_\ell$ and $j_1 i_1 ... \hat{i_k} ... i_\ell$, we get a path. However, this can also be lifted to a path from $i_1 ... i_\ell$ to $j_1 i_1 ... \hat{i_k} ... i_\ell$. Next, by forget j_1 in both $j_1 ... j_\ell$ and $j_1 i_1 ... \hat{i_k} ... i_\ell$, we get a path from the two and hence a path from $i_1 ... i_\ell$ and $j_1 ... j_\ell$.

Case 2: Say $k = \ell$. Then we see $j_1 i_1 ... \hat{i_k} ... i_\ell = j_1 i_1 ... i_{\ell-1}$.

Case 2.1: If $|i_1 - j_1| > 1$. Then we see $i_1 j_1 i_2 ... i_{\ell-1}$ is also a reduced word, but now we can forget i_1 and get a path from $i_1 j_1 ... i_{\ell-1}$ and $i_1 ... i_\ell$. On the other hand, we get a path from $j_1 i_1 ... i_{\ell-1}$ and $j_1 ... j_\ell$ by induction.

Case 2.2: If $|i_1 - j_1| = 1$. Then we see, if we apply Exchange lemma again on $j_1 i_1 ... i_{\ell-1}$ with $i_1 ... i_\ell$, we would get $i_1 j_1 i_1 i_2 ... \hat{i_h} ... i_{\ell-1}$, which is a reduced word with a path to $i_1 ... i_\ell$. But then $i_1 j_1 i_1 i_2 ... \hat{i_h} ... i_{\ell-1}$ has a path to (using braid relation) $j_1 i_1 j_1 ... i_{\ell-1}$, which using induction has a path to $j_1 ... j_\ell$. Hence together we get a path between $i_1 ... i_\ell$ and $j_1 ... j_\ell$. ♡

With this get out of the way, we can now finally define what is Schubert polynomials.

Definition 3.17. The ***ith divided difference*** operator ∂_i on $\mathbb{Z}[x_1, ..., x_n]$ is

$$\partial_i(f) = \frac{f - s_i \cdot f}{x_i - x_{i+1}}$$

We observe:

1. $\partial_i(f)$ is symmetric in x_i and x_{i+1}.
2. If f is symmetric in x_i and x_{i+1} then $\partial_i(f) = 0$.
3. Thus, we see $\partial_i^2(f) = 0$.
4. If f is homogenous of degree d, then $\partial_i(f) = 0$ or $\partial_i(f)$ is homogenous of degree $d - 1$.
5. If $|i - j| \geq 2$ then
$$\partial_j \partial_i(f) = \partial_j(\frac{f - s_i f}{x_i - x_{i+1}}) = \frac{f - s_i f - s_j f + s_j s_i f}{(x_i - x_{i-1})(x_j - x_{j+1})} = \partial_i \partial_j(f)$$
as $s_j s_i = s_i s_j$.
6. If $|i - j| = 1$, then
$$\partial_i \partial_{i+1} \partial_i(f) = \partial_{i+1} \partial_i \partial_{i+1}(f)$$

Definition 3.18. For $w \in S_n$ with reduced word $r = r_1 ... r_{\mathrm{inv}(w)} \in R(w)$, we define

$$\partial_w := \partial_{r_1} \partial_{r_2} ... \partial_{r_{\mathrm{inv}(w)}}$$

Proposition 3.19. *∂_w is well-defined.*

Proof. The $\mathcal{G}(w)$ is connected, hence we are done as they commute when far apart, and braid when they are close. Thus, we can take any reduced words and still get the same ∂_w. ♡

Definition 3.20. For $w_0 = n(n-1)(n-2)...21$, the ***Schubert polynomial*** for w_0 is

$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} ... x_n^0 = \mathfrak{X}_{\text{invcode}(w_0)}$$

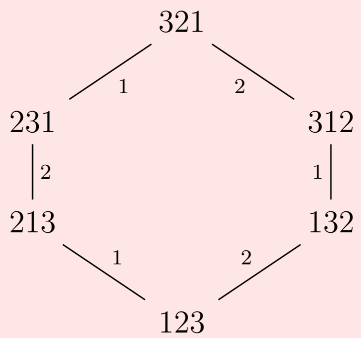
If we have permutation w and suppose we know $\mathfrak{S}_w$ with $w(i) > w(i+1)$, then the ***Schubert polynomial*** for ws_i is given by

$$\mathfrak{S}_{ws_i} = \partial_i(\mathfrak{S}_w)$$

In other word, for any permutation w, we have

$$\mathfrak{S}_w = \partial_{w^{-1}w_0} \mathfrak{S}_{w_0}$$

Example 3.21. Say we have the right weak order on S_3



If we follow the edge, we see $\mathfrak{S}_{321} = x_1^2 x_2$. Hit it with ∂_2, we get $\mathfrak{S}_{312} = x_1^2$. Hit $\mathfrak{S}_{312}$ with ∂_1 we obtain $\mathfrak{S}_{132} = x_1 + x_2$. Finally, $\mathfrak{S}_{123} = 1$. The other side is similar.

Definition 3.22. The ***nilHecke algebra*** N_n has generators $h_1, ..., h_{n-1}$ with relations $h_i h_j = h_j h_i$ for $|i-j| \geq 2$, $h_i h_{i+1} h_i = h_{i+1} h_i h_{i+1}$ and $h_i^2 = 0$.

We observe that:

1. The operators ∂_i gives a representation of N_n on $\mathbb{Z}[x_1, ..., x_n]$.
2. So do $\{\overline{\partial_i}\}$ defined by
$$\overline{\partial}(f) = \partial_i\left((1 + \beta x_{i+1}) f\right)$$
In particular, note $\overline{\partial_w}$ is also well-defined.
3. We can define ***Grothendieck polynomials*** by $\overline{\mathfrak{S}}_w = \overline{\partial_{w^{-1}w_0}} \mathfrak{S}_{w_0}$. In particular, $\overline{\mathfrak{S}_{w_0}} = \mathfrak{S}_{w_0}$.
4. $\mathfrak{S}_w = \overline{\mathfrak{S}}_w(\beta = 0)$.
5. We can also define π_i by $\pi_i(f) = \partial(x_i f)$ is called ***isobaric divided difference*** operators. This no longer give a representation of N_n, but the 0-Hecke algebra H_0 on $\mathbb{Z}[x_1, ..., x_n]$.
6. So do $\overline{\pi_i}$ by
$$\overline{\pi_i}(f) = \pi_i((1 + \beta x_{i+1}) f)$$

7. Set
$$\mathfrak{S}_{w_0}(x_1, ..., x_n; y_1, ..., y_n) = \prod_{i+j \le n} (x - i - y_j)$$
We note we can recover $\mathfrak{S}_{w_0}(x)$ by setting all y in $\mathscr{F}_{w_0}(x_1, ..., x_n, y_1, ..., y_n)$ equal 0. If $w(i) > w(i+1)$ then we inductively define
$$\mathfrak{S}_{ws_i}(x; y) = \partial_i \mathfrak{S}_w(x; y)$$
where our ∂_i consider y as constant. These are ***double Schubert polynomials***.

4 Schubert Calculus

The next topic is Schubert varieties.

Definition 4.1. A ***(complete) flag*** is a chain/nested collection of vector subspaces
$$V_\bullet : 0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq ... \subsetneq V_n = \mathbb{C}^n$$
We use $F\ell_n$ to denote the set of complete flags in $\mathbb{C}^n$.

Clearly for each basis $v_1, ..., v_n$ we get a flag $V_k = \langle v_1, ..., v_n \rangle$. Thus if we take the standard basis, we get the standard flag $\langle e_1, ..., e_n \rangle$. On the other hand, we can also read it backward, i.e. the flag $\langle e_n, e_{n-1}, ..., e_1 \rangle$.

Now we have the space of flags, we want to put some geometry on it. There are few ways to do this.

First, we note $\mathrm{GL}_n(\mathbb{C})$ has an action on $F\ell_n$. We see it only have one orbit. Moreover, if we have $V_\bullet$ and $V_\bullet'$, then $\mathrm{Stab}(V_\bullet')$ is isomorphic to $\mathrm{Stab}(V_\bullet)$ by conjugation.

Definition 4.2. If $B \le \mathrm{GL}_n(\mathbb{C})$ is $\mathrm{Stab}(V_\bullet)$ for some $V_\bullet \in F\ell_n$, then B is called a ***Borel subgroup***.

We note that there is a one to one correspondence between $F\ell_n$ and the set of Borel subgroups.

Now, what is the stablizer of the standard flag E? Well, we want V_0 to be fixed, hence the first column of the matrix to be $(*, 0, ..., 0)^T$. In particular, it is not hard to see , $\mathrm{Stab}(E)$ is the collection of upper triangular invertible matrices. Hence, we get a bijection between $F\ell_n$ and $\mathrm{GL}_n(\mathbb{C})/B$.

Last time, we saw $\mathrm{F}\ell_n$ is in one-to-one correspondence between GL_n/B, where GL_n/B is a quotient of topological spaces, hence we get geometry out of $\mathrm{F}\ell_n$.

This gives $\mathrm{F}\ell_n$ the structure of a smooth projective complex algebraic variety of dimension $\binom{n}{2} = n^2 - \binom{n+1}{2}$. This is exactly the number of inversions of $n(n-1)(n-2)...21$.

Here is another way to think about this variety.

How to specify a flag? Give an ordered basis is the same as give an $n \times n$ invertible matrix, where the basis is just the rows in order from top to bottom.

Clearly this is not unique, so which matrices give the same flag? Well, things that differ by downward row operations, i.e. left action of lower triangular matrices give the same flag. Thus, $\mathrm{F}\ell_n = B_- \backslash \mathrm{GL}_n$, where we act the set of lower triangular matrices on the left of GL_n.

Definition 4.3. A ***Borel subgroup*** of GL_n is a maximal connected solvable subgroup.

Definition 4.4. Let $T \subseteq \mathrm{GL}_n$ be the maximal algebraic torus of diagonal matrices.

We have an action of T on $\mathrm{F}\ell_n$ given by scaling the columns. What are the fixed points? Well, the identity, and actually all permutation matrices, will be fixed by T, and we denote this set by $\{B_{-w}\}_{w \in S_n} = \{e_w\}_{w \in S_n}$.

Also we can have B_+ (invertible upper triangular Borel) act on $\mathrm{F}\ell_n$. Each orbit of this action contains exactly one e_w.

By the ***Bruhat decomposition***, we have

$$\mathrm{GL}_n = \bigsqcup_{w \in S_n} B_- w B_+$$

Viz, GL_n is a disjoint union of $B_- w B_+$, which is a lower triangular times a permutation matrix times a upper triangular.

Thus, if we mod out on the left by B_-, we get the following.

Definition 4.5. We define a ***Schubert cell*** to be

$$\Omega_w^\circ = B_- \backslash B_- w B_+ = e_w B_+$$

Example 4.6. Take $w = \mathrm{Id} = 12345$. Then we see $e_w B_+$ contains matrices with lower triangular part equal 0, 1 on the diagonal, and anything on the upper triangular part. In particular, this means $\Omega_{\mathrm{Id}}^\circ \cong \mathbb{C}^{10}$ as we have dimension equal $\binom{5}{2}$. We note $\mathrm{inv}(w) = 0$ in this case.

Take $w = 54321$. Then e_w looks like

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

But then we don't realy have anything, hence it is just $\mathbb{C}^0$. Note $\mathrm{inv}(w) = 10$ here, hence we sort of have a duality here.

Now take $w = 45132$, with $\operatorname{inv}(w) = 7$. Then

$$\Omega_w^\circ = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

But note since we are moding out B_-, this is basically saying we can mod out anything on the left and down of the 1th in the above, i.e. it is some sort of reflected Rothe diagram. In particular, we have $\Omega_w^\circ \cong \mathbb{C}^3$ as we only have three boxes left.

From the three examples, we see we actually have $\operatorname{codim}(\Omega_w^\circ) = \operatorname{inv}(w)$.

Theorem 4.7. *We have*

$$\operatorname{codim}(\Omega_w^\circ) = \operatorname{inv}(w)$$

$$\Omega_w^\circ \cong \mathbb{C}^{\binom{n}{2} - \operatorname{inv}(w)}$$

Definition 4.8. We define a ***Schubert variety*** as

$$\Omega_w := \overline{\Omega_w^\circ}$$

where the closure is taken in the Zariski topology (or classical topology).

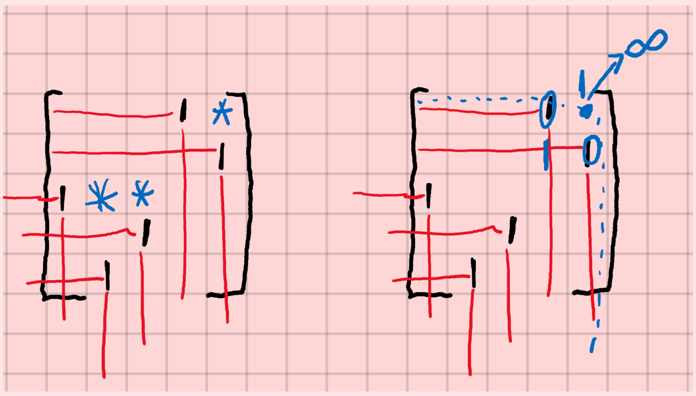
Theorem 4.9. *We have*

$$\Omega_w = \bigsqcup_{w < v} \Omega_v^\circ$$

where $w < v$ is the strong Bruhat order.

Corollary 4.9.1. $\mathrm{F}\ell_n$ *is a complex cell complex.*

Example 4.10. Say we have $w = 45132$. Then we have



where on the left we have Ω_w°, and on the right, we think of the star at the $(1,5)$ position goes to infinity, hence the 1 on the left and down are so small we can just think of it as 0. Then we get a 1 at $(1,5)$ and since we get two more 0, we can move the 1 at $(1,4)$ to $(2,4)$.

Every topological space X has a cohomology ring $H^\star(X) = \bigoplus_i H^i(X)$. In a sense, this measure the number of holes.

Theorem 4.11. *If X has a complex cell structure with n_i cells in (complex) codimension i, then*

$$H^{2i}(X) \cong \mathbb{Z}^{n_i}$$

with a basis indexed by the cells and

$$H^{2i+1}(X) \cong 0$$

Definition 4.12. The $H^\star$ classes of Schubert varieties are called ***Schubert classes*** σ_w and are a basis for $H^\star(\mathrm{F}\ell_n)$.

Theorem 4.13. *There is a dense open subset of $g \in \mathrm{GL}_n$ such that $\sigma_u \cdot \sigma_v$ is equal the $H^\star$ class of $\Omega_u \cap g\Omega_v$.*

In particular, if Ω^v is Ω_v with respect to the opposite flag, then $\sigma_u \cdot \sigma_v$ is equal the class of $\Omega_u \cap \Omega^v$.

Definition 4.14. For $v \in S_n$, we call Σ^v the ***opposite Schubert variety*** and $\Omega_u \cap \Omega^v$ is the ***Richardson variety***.

Theorem 4.15 (Borel)**.**

$$H^\star(\mathrm{F}\ell_n) \cong \mathbb{Z}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n\rangle$$

as graded $\mathbb{Z}$-algebra where e_i are the elementary symmetric functions.

This is $\mathbb{Z}[x_1, \ldots, x_n]$ mod out by the set of symmetric functions without constant terms.

Definition 4.16. We define $\mathbb{Z}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n\rangle$ as the ***coinvariant ring***.

In particular, for Ω_v of codimension k, it should correspond to a coset of homogenous degree k polynomial.

Theorem 4.17. *Under the Borel isomorphism $H^\star(\mathrm{F}\ell_n) \cong \mathbb{Z}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n\rangle$, the Schubert classes correspond to Schubert polynomials. In particular,*

$$\sigma_u \cdot \sigma_v = \sum_{w \in S_n} c_{uv}^w \sigma_w \Leftrightarrow \mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{w \in S_n} c_{uv}^w \mathfrak{S}_w$$

Corollary 4.17.1. *Schubert polynomials has positive structure coefficients. But we do not have a combinatorial rule for the coefficients c_{uv}^w.*

Remark 4.18 (Open Problem)**.** Give a positive combinatorial formula for c_{uv}^w.

Recall $\mathrm{F}\ell_n$ has a action of T on the right given by scaling the columns.

For group G acting on space X, we get not only the cohomology ring, but a G-equivariant cohomology ring $H_G^\star(X)$. So, what is this? Well, the formal definition is quite involved... But broadly speaking, this not only encode the cohomology information, but also the representation information about G.

In particular, $H_T^\star(\mathrm{F}\ell_n)$ is pretty nice, as we have $H_T^\star(\mathrm{F}\ell_n)$ is a free $H_T^\star(\mathrm{pt})$-module, where $H_T^\star(\mathrm{pt})$ is just $\mathbb{Z}[y_1, ..., y_n]$.

Theorem 4.19. *Double Schubert polynomial represent T-equivariant cohomology classes of Schubert varieties. Viz, we have $\sigma_u \circ \sigma_v = \sum_w c_{uv}^w(y)\sigma_w$ iff $\mathfrak{S}_u(x,y) \cdot \mathfrak{S}_v(x,y) = \sum_w c_{uv}^w(y)\mathfrak{S}_w(x,y)$, where now the coefficients are polynomial in y.*

We also have a K-theory ring $K(\mathrm{F}\ell_n)$, where elements are equivalence classes of algebraic vector bundles on $\mathrm{F}\ell_n$. For each Ω_w, we get a class $[\mathscr{O}_{\Omega_w}]$ in $K(\mathrm{F}\ell_n)$ for its structure sheaf $\mathscr{O}_{\Omega_w}$, which is a linear combination of vector bundles.

There are still a basis, and the structure coefficient in this case is controled by Grothendieck polynomials.

Theorem 4.20. *Grothendieck polynomials represent K-classes. Viz, $[\mathscr{O}_{\Omega_u}] \cdot [\mathscr{O}_{\Omega_v}] = \sum_w K_{uv}^w [\mathscr{O}_{\Omega_w}]$ iff $\overline{\mathfrak{S}_u}(\beta = -1) \cdot \overline{\mathfrak{S}_v}(\beta = -1) = \sum_w K_{uv}^w \overline{\mathfrak{S}_w}(\beta = -1)$.*

Now we get back to combinatorics about Schubert polynomials. The next formula gives a way to compute Schubert polynomials.

Definition 4.21. Let α, β be strong compositions, then we say β is ***α-compatible*** if:

1. $\ell(\alpha) = \ell(\beta)$
2. β is weakly increasing ($\beta_i \le \beta_{i+1}$)
3. β is bounded above by α ($\beta_i \le \alpha_i$)
4. β strictly increasing whenever α does (if $\alpha_i < \alpha_{i+1}$ then $\beta_i < \beta_{i+1}$)

In this case, we write $\beta \looparrowright \alpha$.

Theorem 4.22 (Billey-Jockusch-Stanley, 1993)**.**

$$\{\mathfrak{S}_w\} \to \{\mathfrak{X}_a\}$$

In particular, we have

$$\mathfrak{S}_w = \sum_{\alpha \in R(w)} \sum_{\beta \looparrowright \alpha} \prod_{i=1}^{n} x_{\beta_i}$$

where $\beta \looparrowright \alpha$ means β is α-compatible.

Example 4.23. Say $w = 321$, $\mathfrak{S}_w = x_1^2 x_2 = \mathfrak{X}_{210}$. Then we have $R(w) = \{121, 212\}$.

Thus, say $\alpha = 212$, then there is only one compatible β, which is $\beta = 112$.

Next, say $\alpha = 121$. Then $\beta = \varnothing$. Hence

$$\mathfrak{S}_w = \prod_{\beta \in \{112\}} x_{\beta_i} = x_1 x_1 x_2$$

Definition 4.24. For vector space $U \subseteq V$ with bases B_w, B_v, if $B_u \subseteq B_v$, then we say B_v ***lifts*** B_u to V, or B_v is a ***lift*** of B_u.

Theorem 4.25. *$\{\mathfrak{S}_u\}$ lift $\{s_\lambda\}$ from* Sym *to* ASym*. To realize s_λ as a Schubert polynomial, add 0's to the end of λ to make it length n, then reverse to get Lehmer code of a permutation w_λ. Then $\mathfrak{S}_{w_\lambda} = s_\lambda(x_1, \ldots, x_n)$.*

Example 4.26. Let $s_{21} \in \mathrm{Sym}_3$. Then $\lambda = 210$ and reverse it we get 012, which is going to be the Lehmer code of a permutation. This permutation w_λ here is 13524

Definition 4.27. Permutations of the form w_λ are called ***Grassmannian*** permutations.

Definition 4.28. The ***Grassmannian*** $\mathrm{Grass}_k(\mathbb{C}^n)$ is the parameter space for k-dimensional vector subspaces of $\mathbb{C}^n$.

It turns out $H^\star(\mathrm{Grass}_k(\mathbb{C}^n)) = \mathrm{Sym}_n$ with a Schubert basis of Schur polynomials. The relation between $\mathrm{F}\ell_n$ and $\mathrm{Grass}_k(\mathbb{C}^n)$ is that, we have a forgetful map

$$\mathrm{F}\ell_n \to \mathrm{Grass}_k(\mathbb{C}^n)$$

and hence it induces a (injective) map between cohomologies

$$H^\star(\mathrm{F}\ell_n) \hookleftarrow H^\star(\mathrm{Grass}_k(\mathbb{C}^n))$$

Those map exactly takes σ_λ to σ_{w_λ} and s_λ to $\mathfrak{S}_{w_\lambda}$.

Next, we note BJS (Billey-Jockusch-Stanley) makes us want to define for composition α, a polynomial

$$\mathfrak{F}(\alpha) = \sum_{\beta \looparrowright \alpha} \prod_{i=1}^{n} x_{\beta_i}$$

This might look like a silly thing to do, but it is actually going to be useful (as it is going to be a basis), except that this is the wrong indexing.

Well, some of the $\mathfrak{F}(\alpha)$ are bad, for example, we already computed $\mathfrak{F}(121)$, which sums over the empty set and hence $\mathfrak{F}(121) = 0$.

Definition 4.29. For composition α, if $\mathfrak{F}(\alpha) \neq 0$, then $\mathfrak{F}(\alpha)$ is called a ***fundamental slide polynomial*** (due to Assaf-Searles, 2017).

Besides the problem of many of $\mathfrak{F}(\alpha)$ are zero, we also have the problem that many $\mathfrak{F}(\alpha)$ are equal. For example, $\mathfrak{F}(212) = \mathfrak{X}_{210} = \mathfrak{F}(312)$. All those suggest we have wrong index of $\mathfrak{F}$.

Now, observe if α has a sequence of compatible compositions, it has one that is termwise maximal element, which we will denote by $\hat{\beta}(\alpha)$ (so basically we start at the right-most location, and only decrease one if we have to, this gives the maximal element).

Lemma 4.30. *If $\hat{\beta}(\alpha) = \hat{\beta}(\gamma)$ then $\mathfrak{F}(\alpha) = \mathfrak{F}(\gamma)$.*

Proof. From $\hat{\beta}(\alpha)$ we can determine all other compatible compositions. ♡

Definition 4.31. We define $a(\alpha)$ by $a(\alpha_i)$ equal the number of i in $\hat{\beta}(\alpha)$.

Example 4.32. We see $a(212) = 210 = a(312)$, as $\hat{\beta}(212) = \hat{\beta} = 112$.

This will be the way we index things, and hence we get the following definition.

Definition 4.33. We define $F_{a(\alpha)} = F(\alpha)$ for any weak composition $a(\alpha)$.

This is a little bit weird, as we don't know yet every weak composition is of the form $a(\alpha)$.

Lemma 4.34. *Every weak composition a is of the form $a(\alpha)$ for some α.*

Proof. From a, set $\alpha = 1...12...23...34...$ where we have a_1 many 1, a_2 many 2, and so on. Then $\hat{\beta}(\alpha) = \alpha$ and so $a(\alpha) = a$ as desired. ♡

Lemma 4.35. *If $a \neq b$, then $\mathfrak{F}_a \neq \mathfrak{F}_b$.*

Proof. Say $a = a(\alpha)$ and $b = b(\beta)$ as constructed in above lemma. Then $\alpha \neq \beta$ but by the way the above construction works, $\hat{\beta}(\alpha) = \alpha$ and $\hat{\beta}(\beta) = \beta$, i.e. we have distinct termwise maxima so $\mathfrak{F}_\alpha \neq \mathfrak{F}_\beta$ as desired. ♡

Example 4.36. We have $\mathfrak{F}_{0102} = \mathfrak{F}(244)$. Then the compatible sequence is given by $\beta = (244, 234, 233, 144, 134, 133, 124, 123, 122)$ and hence

$$\mathfrak{F}(244) = x_2x_4x_4 + x_2x_3x_4 + x_2x_3x_3 + x_1x_4x_4 + \dots$$

Definition 4.37. Let b be weak composition, then b is a ***slide*** of weak composition a if b is obtained from a by a sequence of the following local moves:

1. $0p \Rightarrow p0$ where p is positive number, i.e. we can slide positive numbers to the left.
2. $0p \Rightarrow qr$ for $q + r = p$ and $q, r > 0$, i.e. we can chop positive numbers into pieces.

Theorem 4.38.

$$\mathfrak{F}_a = \sum_{b \text{ slide of } a} x^b$$

Lemma 4.39. *Take reverse lexicographic order on monomials (i.e. $x^a < x^b$ if $b - a$ has rightmost non-zero entry to be positive). Then the leading term of $\mathfrak{F}_a$ is x^a.*

Corollary 4.39.1. $\{\mathfrak{F}_a\}$ *is a basis of* ASym.

Proof. Suppose $c_1\mathfrak{F}_{a_1} + \ldots + c_k\mathfrak{F}_{a_k} = 0$. One a_i is biggest in our total order, its leading term appears nowhere else. Thus c_i must be zero. Now induction follows to conclude $c_1 = \ldots = c_k = 0$. This concludes linearly independent, and it is clearly span as the number of weak compositions is the right dimension of ASym, hence it must span as well. ♡

Definition 4.40. Say a ***dominates*** b and write $a \geq b$ if for all k we have $\sum_{i=1}^k a_i \geq \sum_{i=1}^k b_i$.

We note this is just the same as partitions, and in particular this is also not a total order, e.g. 2220 and 3111 are not comparable.

Theorem 4.41.

$$\mathfrak{F}_a = \sum_{\substack{b^+ \vDash a^+ \\ b \geq a}} x^b$$

This look similar. Now, we ask, when is $\mathfrak{F}_a \in \mathrm{QSym}$? If we look back to the example from last lecture, we see we want those weak compositions that shoved all the zeros to the left.

Definition 4.42. Say weak composition a is ***quasistrong*** if its non-zero terms appear consecutively.

Lemma 4.43. *If a is not quasistrong, then* $\mathfrak{F}_a \notin \mathrm{QSym}_n$.

Proof. Choose $i < j < k$ with $a_i > 0, a_j = 0, a_k > 0$. Then define new weak composition b by

$$b_h = \begin{cases} 0, & h = 1 \\ a_{h-1}, & 2 \leq h \leq j \\ a_h, & h > j \end{cases}$$

Then $x^a \in \mathfrak{F}_a$ but $x^b \notin \mathfrak{F}_a$ (because we literally slide things in the wrong way), thus it is not quasisymmetric, i.e. we found a explicit missing term in $\mathfrak{F}_a$ to make it not quasisymmetric. ♡

Proposition 4.44. *If $a = 0^k\alpha$ is quasistrong, then $\mathfrak{F}_a \in \mathrm{QSym}_{\ell(a)}$ and $\mathfrak{F}_a = F_\alpha(x_1, \ldots, x_{\ell(a)})$.*

Proof.

$$\mathfrak{F}_a = \sum_{\substack{b^+ \vDash a^+ \\ b \geq a}} x^b = \sum_{\substack{b^+ \vDash \alpha \\ b \geq a}} x^b = F_\alpha(x_1, \ldots, x_{\ell(a)})$$

♡

Corollary 4.44.1. *$\mathfrak{F}_a$ lifts F_α basis from* QSym *to* ASym.

Theorem 4.45. *$\{\mathfrak{F}_a\}$ has positive structure coefficients, with an explicit positive combinatorial rule that extending the formula for F_α.*

Proof. Will follow from more general K-theory analogue. ♡

Definition 4.46. A ***weak komposition*** is a weak composition where the nonzero terms are coloured either red or black.

Definition 4.47. Weak komposition b is a ***glide*** of a weak composition a if it can be obtained by a sequence of following locally:

1. $0p \Rightarrow p0$ if $p > 0$
2. $0p \Rightarrow qr$ if $q + r = p$ and $p, q, r > 0$.
3. $0p \Rightarrow s\color{red}{t}$ if $s, t > 0$ and $s + t = p + 1$.

where the black letters are actually black in the definition of komposition (i.e. we cannot use 1, 2 to red $\color{red}{0p}$).

Example 4.48. Say $a = 0200201$ is a weak komposition. Then glides includes

$$1\color{red}{2}\color{black}{0201}\color{red}{1}$$

$$21\color{red}{211}\color{black}{10}$$

Definition 4.49. The ***fundamental glide polynomial*** $\overline{\mathfrak{F}_a}$ is

$$\overline{\mathfrak{F}_a} = \sum_{b \text{ glide of } a} \beta^{\#\mathrm{red}} x^b$$

where here $x^b = x_1^{b_1} \ldots x_n^{b_n}$ regardless of the colour.

Example 4.50. Say $a = 0201$, then we can get the following glides

$$0210, 2001, 1101, 021\color{red}{1}\color{black}{}, 2\color{red}{1}\color{black}{01}, 1\color{red}{2}\color{black}{01}$$

$$2010, 1110, 201\color{red}{1}\color{black}{}, 111\color{red}{1}\color{black}{}, 2\color{red}{11}\color{black}{1}, 2\color{red}{1}\color{black}{10}, 1\color{red}{2}\color{black}{1}\color{red}{1}$$

$$2100, 21\color{red}{1}\color{black}{0}, 210\color{red}{1}\color{black}{}, 21\color{red}{11}$$

Then the fundamental glide polynomial is given by

$$\begin{aligned}\overline{\mathfrak{F}_a} = x^{0210} + x^{0210} + x^{2001} + x^{1101} + x^{1110} + x^{2010} + x^{2100}\\ + \beta(x^{0211} + 2x^{2101} + x^{1201} + x^{1111} + x^{2011} + 2x^{2110} + x^{1210})\\ + \beta^2(x^{1211} + 2x^{2111})\end{aligned}$$

A general way to know if we are doing it right is that if we set $x = 1$ and $\beta = -1$, then we should end up with 1. This is related to the Euler's characteristic of some simplicial complex.

We also observe that $\overline{\mathfrak{F}_a}(\beta = 0) = \mathfrak{F}_a$.

Definition 4.51. We define ***multi-fundamental quasisymmetric polynomial*** $\overline{F_\alpha}(x_1, ..., x_n) := \overline{\mathfrak{F}_{0^{n-\ell(\alpha)}\alpha}}$. This definition is due to Lam-Pylyavskyy in 2007.

Theorem 4.52. *$\{\overline{\mathfrak{F}_a}\}$ has positive structure coefficients. Moreover, there is an explicit positive combinatorial rule, such that specialized to $\{\mathfrak{F}_a\}, \{F_\alpha\}, \{\overline{F_\alpha}\}$.*

The proof is actually not so bad, once we write down explicitly what the rule is. And we already know a combinatorial rule for F_α, thus, it is natural to consider shuffles of weak compositions.

Recall if A, B are words on disjoint alphabets $\mathcal{A}, \mathcal{B}$, the ***shuffle product*** $\mathcal{A} ⧢ \mathcal{B}$ is all permutations of AB with the subwords on $\mathcal{A}$ is A and the subwords on $\mathcal{B}$ is B.

Definition 4.53. For alphabet $\mathcal{A}$, let $\mathcal{A}^{gen} = \{i^j : i \in \mathcal{A}, j \in \mathbb{Z}_+\}$. For A a word in $\mathcal{A}$, we define A^{gen} by replacing jth instance of i with i^j from left to right.

Example 4.54. If $A = 1311212$, then $A^{gen} = 1^1 3^1 1^2 1^3 2^1 1^4 2^2$.

Definition 4.55. The ***genomic shuffle product*** $A ⧢_{gen} B$ is all words in $(\mathcal{A} \mathbin{\dot\cup} \mathcal{B})^{gen}$ such that:

1. If i^j left of i^k, then $j \leq k$.
2. No consecutive i^j.
3. Every genotype is in $A ⧢ B$, where ***genotype*** means we delete all but one of each symbol, forget the superscript.

Example 4.56. The genomic shuffle $331 ⧢_{gen} 62$ is an infinite set. Indeed, consider $3^1 3^2 6^1 1^1 2^1 1^1 2^1 \overline{1^1 2^1}$, i.e. we can put infinitely many $1^1 2^1$ at the end and they are all valid shuffles. However, we note this contains finitely many words of each length.

For length 5 element, we have 10 words, and it is basically $331 ⧢ 62$.

For length 6 element, we have 35 words. For example we have $6^1 3^1 2^1 3^1 3^2 1^1$, then genotypes for this includes $63231, 62331, 36231$. We also have $3^1 6^1 2^1 3^2 1^1 2^1$, and a genotype is 36312.

Remark 4.57 (Open Question?). What is

$$\sum_{w \in A \sqcup\!\sqcup_{gen} B} x^{\ell(w)}$$

Definition 4.58. Let $S = (w_1, ..., w_n)$ be a sequence of words in $\mathcal{A}$ and $\mathcal{B} \subseteq \mathcal{A}$. Then $\mathrm{Comp}_{\mathcal{B}}(S) = (a_1, ..., a_n)$ with a_i equal number of $\mathcal{B}$-words/letters in w_i. We also let $\mathrm{Comp}(S) = \mathrm{Comp}_{\mathcal{A}}(S)$.

Definition 4.59. Put lexicographic order on $\mathbb{Z}_+^{gen}$ with $i^j < k^l$ if $i < k$ or $i = k$ and $j < l$. For C a word in $\mathbb{Z}_+^{gen}$, let $\mathrm{Runs}(C)$ be the sequence of maximal increasing runs.

Example 4.60. For example, if $C = 3^1 3^2 6^1 1^1 2^1 1^1 2^1 \overline{1^1 2^1}$ then $\mathrm{Runs}(C) = 3^1 3^2 6^1 | 1^1 2^1 | 1^1 2^1 | \overline{1^1 2^1}$

Let us spit out how to multiply fundamental glide polynomials. Consider two set of letters $\mathcal{A}, \mathcal{B}$ with $\mathcal{A} \cap \mathcal{B} = \varnothing$. Then $A \sqcup\!\sqcup B$ is the set of all permutations with subword from $\mathcal{A}$ being A and subword from $\mathcal{B}$ being B.

Then, we upgraded this to $\mathcal{A}^{gen}$, the genomic version. Then for A in $\mathcal{A}$, we have A^{gen} by ith instance of j becomes j^i.

Using this, we defined genomic shuffle product last time and worked out some examples.

Definition 4.61. A ***genetype*** of $\mathrm{Runs}(C)$ means delete all but one of each symbol and forget the superscript.

Example 4.62. Let $C = 6^1 3^1 6^1 3^2 1^1 2^1$, then $\mathrm{Runs}(C) = (6^1, 3^1 6^2, 3^2, 1^1, 2^1)$, $\mathrm{Comp}(\mathrm{Runs}(C)) = (1, 2, 1, 2)$. A genetype of $\mathrm{Runs}(C)$ would be $G_1 = (6, 3, 3, 1, 2)$ and $G_2 = (\varnothing, 36, 3, 12)$. Then $\mathrm{Comp}_{even}(G_2) = (0, 1, 0, 1)$.

Definition 4.63. Let a, b be weak compositions of length n. Define

$$A = (2n-1)^{a_1} ... (3)^{a_{n-1}} (1)^{a_n}$$

$$B = (2n)^{b_1} ... (4)^{b_{n-1}} (2)^{b_n}$$

The ***genomic shuffle set*** $\mathrm{GSS}(a, b)$ to be all $C \in A \sqcup\!\sqcup_{gen} B$ such that for all genetype of $\mathrm{Runs}(C)$, $\mathrm{Comp}_{odd}(G) \geq a$ and $\mathrm{Comp}_{even}(G) \geq b$.

We note by require $\mathrm{Comp}_{odd}(G)$ and $\mathrm{Comp}_{even}(G)$ dominate some composition, we make sure those infinite terms vanishes.

Example 4.64. Suppose $a = 021, b = 101$. Then $A = 331$ and $B = 62$ and $\mathrm{GSS}(a,b)$ is equal the following

$$\mathrm{GSS}(a,b)_5 = \{6^1 2^1 3^1 3^2 1^1, 6^1 3^1 3^2 1^1 2^1, 3^1 6^1 2^1 3^2 1^1, 3^1 6^1 3^2 1^1 2^1, 3^1 3^2 6^1 2^1 1^1, 3^1 3^2 6^1 1^1 2^1\}$$

$$\mathrm{GSS}(a,b)_6 = \{6^1 2^1 3^1 3^2 1^1 2^1, 3^1 6^1 2^1 3^1 3^2 1^1, \ldots\}$$

where the subscript means the length of the shuffle words. We have in total 10 shuffles for length 5, but only 6 satisfy the dominate condition. There are in total 35 shuffles of length 6, but only 8 dominates. For length 7, we have in total 81 shuffles, but only 3 works

$$\mathrm{GSS}_7(a,b) = \{3^1 6^1 2^1 3^1 3^2 1^1 2^1, 3^1 3^2 6^1 2^1 3^2 1^1 2^1, 3^1 3^2 6^1 1^1 2^1 1^1 2^1\}$$

Definition 4.65. Let a, b be two weak compositions of length n. For $C \in \mathrm{GSS}(a,b)$, let $\mathrm{Bump\,Runs}(C)$ be the dominance minimal way to insert empty runs to $\mathrm{Runs}(C)$ maintaining the dominance condition.

Definition 4.66. The ***glide product*** $a \sqcup\!\sqcup_{gen} b$ is the multiset $\{\{\mathrm{Comp}(\mathrm{Bump\,Runs}(C)) : C \in \mathrm{GSS}(a,b)\}\}$.

Theorem 4.67.

$$\overline{\mathfrak{F}_a} \cdot \overline{\mathfrak{F}_b} = \sum_c \beta^{|c|-|a|-|b|} g_{ab}^c \overline{\mathfrak{F}_c}$$

where g_{ab}^c is the multiplicity of c in $a \sqcup\!\sqcup_{gen} b$.

Proof. WLOG set $\beta = 1$. Given $C \in \mathrm{GSS}(a,b)$, form $\overline{C}$ by inserting red $|$ to separate as in Bump Runs. For example, we would have $3^1 3^2 6^1 1^1 2^1$ becomes $3^1 3^2 6^1 || 1^1 2^1$. Let $\mathrm{shift}\,\overline{C}$ be the words obtainable from $\overline{C}$ by:

1. duplicating any i^j.
2. moving $|$ to the right.

such that if i^j followed by k^l then $i^j < k^l$.

For example, if $\overline{C} = 3^1 3^2 6^1 || 1^1 2^1$, then

$$\mathrm{shift}\,\overline{C} = \begin{cases} 3^1 3^2 6^1 || 1^1 2^1 \\ 3^1 3^2 6^1 | 1^1 | 2^1 \\ 3^1 3^2 6^1 | 1^1 2^1 | \\ 3^1 3^2 6^1 | 1^1 | 1^1 2^1 \\ 3^1 3^2 6^1 | 1^1 2^1 | 2^1 \end{cases}$$

Let $\overline{\mathrm{GSS}(a,b)} = \bigcup_{C \in \mathrm{GSS}(a,b)} \mathrm{shift}(\overline{C})$.

Let $M(a,b) = \{(a', b')\}$ be the set of tuples of weak compositions such that a' is a glide of a and b' is a glide of b. These correspond to the monomials in $\overline{\mathfrak{F}_a}(\beta = 1) \cdot \overline{\mathfrak{F}_b}(\beta = 1)$.

We claim $M(a,b)$ is in bijection with $\overline{\mathrm{GSS}(a,b)}$.

For $D \in \overline{\mathrm{GSS}(a,b)}$, let $\mathrm{Seq}(D)$ be the sequence of words between bars. We get $(a',b') \in M(a,b)$ by

$$(a',b') = (\mathrm{Comp}_{odd}(\mathrm{Seq}(D)), \mathrm{Comp}_{even}(\mathrm{Seq}(D)))$$

where a'_i is red if the ith part of $\mathrm{Seq}(D)$ contains a odd^{gen} letter from earlier in the sequence.

For example, take $D := 3^1 3^2 6^1 {\color{red}||} 1^1 2^1 \in \mathrm{shift}(\overline{C})$, then $\mathrm{Seq}(D) = (3^1 3^2 6^1, \varnothing, 1^1 2^1)$ and hence $(a',b') = ((2,0,1),(1,0,1))$ and we have no red because we have no repeat. Now take $D = 3^1 3^2 6^1 {\color{red}|} 1^1 {\color{red}|} 1^1 2^1$, then $\mathrm{Seq}(D) = (3^1 3^2 6^1, 1^1, 1^1 2^1)$. This gives $((2,1,1),(1,0,1))$ and the second one should be red, i.e.

$$(a',b') = ((2,1,{\color{red}1}),(1,0,1))$$

Now consider $(a',b') \in M(a,b)$, we recover $D \in \overline{\mathrm{GSS}(a,b)}$ by:

1. The first run of D is the first a'_1 letters of A^{gen} and the first b'_1 letters of B^{gen}, sorted to be increasing.
2. The second run of D is the next a'_2 letters of A^{gen} and b'_2 letters of B^{gen}, sorted to be increasing, EXCEPT, if a'_2 or b'_2 is red, then repeat the last letter of the appropriate alphabet from the previous part.
3. The rest are similar to the second run.

Let us look at an example. Note in the running example, we have $A^{gen} = 3^1 3^2 1^1$, $B^{gen} = 6^1 2^1$. Then $(210, 11{\color{red}1}) \in M(a,b)$. This would maps to

$$3^1 3^2 6^1 {\color{red}|} 1^1 2^1 {\color{red}|} 2^1$$

These maps are inverse of each other if you think about it...

So, $\overline{\mathrm{GSS}(a,b)}$ bijects with monomials of $\overline{\mathfrak{F}}_a(\beta = 1) \cdot \overline{\mathfrak{F}}_b(\beta = 1)$. But for $C \in \mathrm{GSS}(a,b)$, the monomials for $\mathrm{shift}(\overline{C})$ make up $\overline{\mathfrak{F}}_{\mathrm{Comp}(\mathrm{Bump\,Runs}(C))}(\beta = 1)$.

For example, if $\overline{C} = 3^1 3^2 6^1 {\color{red}||} 1^1 2^1$, then we get monomials $x^{302} + x^{311} + x^{320} + x^{312} + x^{321}$ which equals $\overline{\mathfrak{F}}_{302}(\beta = 1)$.

Thus, $\overline{\mathrm{GSS}(a,b)}$ are partitioned by $C \in \mathrm{GSS}(a,b)$ with each block giving the appropriate glide polynomial. ♡

The reason why we cannot do this to Schubert polynomial is that, we don't have a good partitioning like GSS above for Schubert polynomial, i.e. we don't know how to collect monomials into Schubert polynomials.

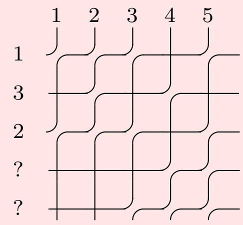
> **Remark 4.68** (Open Problem)**.** What is the geometric interpretation of glide polynomials?

5 Pipe Dreams

Now we dealt with positive formula for glide, we talk about a (non-positive) formula for Schubert polynomial. This lead to pipe dreams.

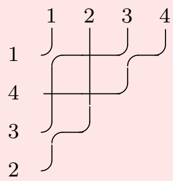
Definition 5.1. A (reduced) pipe dream is a tiling of the plane using two blocks $\boxplus$-block and -block (called "cross" and "jr/bump") with finitely many cross blocks and no 2 pipes cross more than once.

Example 5.2. Here is a (partial) example that's not reduced:



Let us see another partial example (we say partial because those pipe dream suppsed to be infinite).

Example 5.3. Consider



where we ignored some of the bump blocks. We note we can associate the above reduced pipe dream to a permutation, by put $1, 2, 3, 4$ above, and see where the pipe lead us to at the end.

Definition 5.4. We let $PD_0(w)$ be the set of reduced pipe dreams for permutation w.

Theorem 5.5 (Fomin0Kirillov, Bergeron-Billey,Knutson-Miller)**.**

$$\mathfrak{S}_w(x) = \sum_{P \in PD_0(w)} x^{\mathrm{wt}(P)}$$

$$\mathfrak{S}_w(x, y) = \sum_{P \in PD_0(w)} \prod_{(i,j) \text{ with } \boxplus} (x_i - y_j)$$

where wt *means the number of* $\boxplus$*-blocks in each row.*

Example 5.6. Let $w = 15324$. Then we have the following reduced pipe dreams (in the matrix, we use + to denote the cross and empty to mean bump):

$$\begin{array}{|lll} \hline + & + & + \\ + & & \end{array} \qquad \begin{array}{|llll} \hline & & & + \\ + & + & & \\ + & & & \end{array} \qquad \begin{array}{|lll} \hline + & + & + \\ + & & \end{array}$$

$$\begin{array}{|cccc} & & + & + \\ + & & & \\ + & & & \end{array} \quad \begin{array}{|cccc} & + & + & + \\ & & & \\ + & & & \end{array} \quad \begin{array}{|cccc} & + & & \\ + & + & + & \end{array} \quad \begin{array}{|cccc} & + & & + \\ + & + & & \end{array}$$

The reduced words for w is $4323, 4232, 2432$. The compatible sequences are given by

$$4323 : 1112, 1113, 1123, 1223, 2223$$

$$4232 : 1122$$

$$2432 : 1222$$

This is a good sign as we have seven pipe dreams and it matchs up.

Next, we see 1222 would correspond to

$$\begin{array}{|cccc} & + & & \\ + & + & + & \end{array}$$

because it has one cross in the first row, and the three cross in the second row. Similarly, we see all of the weights matche up exactly.

This is because, when we view the pipe dreams, we also recorded the reduced word. The way to read reduced word from pipe dreams is that, we read the crosses from top to bottom, right to left, and record the reversed diagonal it is located at.

Lemma 5.7. *Let P be a reduced pipe dream. If all $\boxplus$-blocks in row i are strictly east of all $\boxplus$-blocks in row $i+1$. Then slide them diagonally southwest, the resulting diagram is a pipe dream for the same permutation and same reduced word.*

Proof. Each $\boxplus$-block stay at the same diagonal and we read in the same order. ♡

Definition 5.8. The ***destandardization*** of $P \in PD_0(w)$ is the pipe dream obtained by applying the lemma as much as possible.

Definition 5.9. A pipe dream is ***quasi-Yamanouchi*** if the leftmost $\boxplus$-block in each row satisfies either:

1. it is in the first column, or
2. it is weakly west of some $\boxplus$-block in the row below.

We use $QPD_0(w)$ to denote the set of quasi-Yamanouchi pipe dreams for w.

Lemma 5.10. *A pipe dream is destandardization iff it is quasi-Yamanouchi.*

Theorem 5.11.

$$\mathfrak{S}_w = \sum_{P \in QPD_0(w)} \mathfrak{F}_{wt(P)}$$

Proof. For $P \in QPD_0(w)$, let $dst^{-1}(P)$ be the set of pipe dreams that destandardization to P. Enough to show

$$\sum_{Q \in dst^{-1}(P)} x^{\mathrm{wt}(Q)} = \mathfrak{F}_{\mathrm{wt}(P)}$$

If $dst(Q) = P$, then $\mathrm{wt}(Q)$ is a slide of $\mathrm{wt}(P)$. Conversely, if a is a slide of $\mathrm{wt}(P)$, then there is a unique pipe dream Q correspond to a that its destandardization is P. To get this Q, we see how to get a from $\mathrm{wt}(P)$, but then this is a sequence of local action, which can be translated to the local move of P. ♡

6 Kohnert's Formula

Next, we introduce two more rules for Schubert polynomials. The first one is Kohnert's formula, which is conjectured in 1990 by Kohnert. This is "proved" Winkel in 1999, but it has a bunch of gaps (he traced through the divided difference operator this time). Next, in 2002, Winkel proposed another proof uses some recurrence, but there are also gaps and the details become hard. In 2017, Assaf proposed another proof, but the detail also become devilish.

It also have a Grothendieck extension, which is conjectured by Ross and Yong in 2015.

Definition 6.1. Let D be a collection of boxes in one quadrant. A ***Kohnert move*** is to take the rightmost box in any row and move it stright up to the next available open spot.

We let $KD(D)$ be the set of all diagrams reachable from D by a sequence of Kohnert moves.

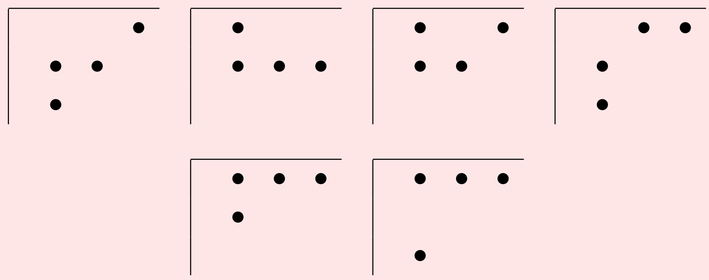
Then we define $\mathrm{wt}(D) = (a_1, ..., a_n)$ where a_i is the number of boxes in row i.

Example 6.2. Suppose $D = D(15324)$, i.e. we have D is the Rothe diagram of 15324, and it looks like

$$\begin{array}{|ccc}\hline & & \\ \bullet & \bullet & \bullet \\ \bullet & & \end{array}$$

where the $\bullet$'s denotes boxes. Then, we get the following configurations from the

Kohnert moves:



Include $D = D(15324)$, we indeed get 7 diagrams, and the weight match up.

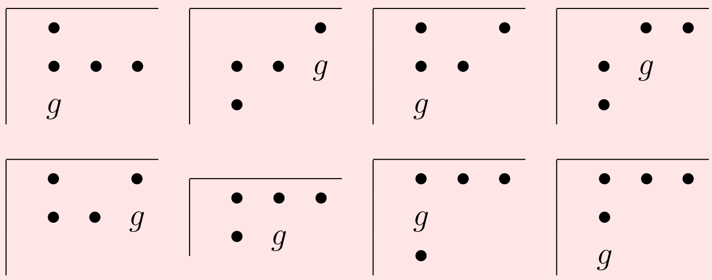
Conjecture 6.3 (Kohnert's Formula)**.**

$$\mathfrak{S}_w = \sum_{D \in KD(D(w))} x^{\mathrm{wt}(D)}$$

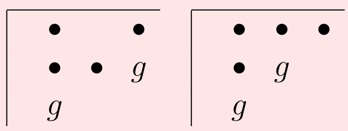
Definition 6.4. A ***spectral Kohnert move*** takes the rightmost box in some row and moves it stright up to the next available open spot, leaving a "ghost" in the original position. Neither oridinary nor spectral Kohnert moves can pass through a ghost, and a ghost cannot move.

We use $\overline{KD}(D)$ to denote all spectral diagrams reachable from D using both Kohnert move and spectral Kohnert move.

Example 6.5. Still take $D = D(15324)$. Then some example of spectral Kohnert moves are (we use letter g to denote a ghost)



We also have some diagrams with two ghosts:



Conjecture 6.6 (Ross-Yong)**.**

$$\overline{\mathfrak{S}}_w = \sum_{D \in \overline{KD}(D(w))} \beta^{\#\text{ of ghosts}} x^{\mathrm{wt}(D)}$$

The problem for those Kohnert moves is that, we do not have double version of Kohnert diagrams. In particular, this means our geometry would be realy weird in the sense that it would not be nice enough to give double version, but it is good enough to spit out the single version.

7 Demazure Characters

Last time we had Kohnert's formula for the Schubert polynomial, which is take a permutation w, convert to Rothe diagram, then take all Kohnert moves. This is not obvious at all, and today we are going to do something more obvious.

For a weak composition a, let $D(a)$ be Young diagram, i.e. a_i left-justified boxes in row i.

Definition 7.1. The ***Demazure character/key polynomial*** is

$$\mathfrak{D}_a := \sum_{D \in KD(D(a))} x^{\mathrm{wt}(D)}$$

Conjecture 7.2. The Lascoux polynomial is given by

$$\overline{\mathfrak{D}}_a = \sum_{D \in \overline{KD}(D(a))} \beta^{\#\text{ of ghosts}} x^{\mathrm{wt}(D)}$$

Example 7.3. Suppose $a = \lambda$, then there are no Kohnert moves and it just become x^λ.

In the homework, we are asked to describe Rothe diagrams for 2143-avoiding permutation.

Proposition 7.4. *A permutation w is 2143-avoiding if and only if $\mathfrak{F}_w = \mathfrak{D}_{\mathrm{invcode}(w)}$.*

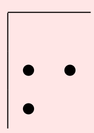
Recall $\pi_i(f) = \partial_i(x_i f)$ and $\overline{\pi_i}(f) = \pi_i((1+\beta x_{i+1})f)$. They both satisfy braid relation.

Let as_i be a with ith and $(i+1)$st entries swaped. Let $w(a)$ be the shortest permutation with $a \cdot (w(a)) = \mathrm{sort}(a)$, where for example $\mathrm{sort}(3412) = 4321$.

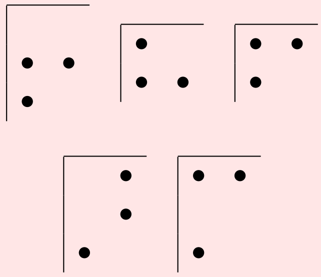
Definition 7.5. $\mathfrak{D}_a = \pi_{w(a)} x^{\mathrm{sort}(a)}$ and $\overline{\mathfrak{D}}_a = \overline{\pi}_{w(a)} x^{\mathrm{sort}(a)}$.

Example 7.6. In the silly case, if $a = \lambda = \mathrm{sort}(a)$, then $\overline{\mathfrak{D}}_a = \mathfrak{D}_a = x^{\mathrm{sort}(a)}$.

Next we consider a real example. If $a = 021$, then we get



Enumerate over all Kohnert moves, we get



This gives weights $021, 120, 210, 111, 201$.

On the other hand, we see $a = 021$ and so $w(a) = s_1 s_2$. This gives

$$\pi_{w(a)} x^{\mathrm{sort}(a)} = \pi_1(\pi_2(x^{210})) = (x_1^2 x_2 + x_1 x_2^2) + (x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3)$$

after a lengthy computation. Clearly we end up with the same equation.

Theorem 7.7. *The* $\{\mathfrak{D}_a\}$ *and* $\{\overline{\mathfrak{D}}_a\}$ *are bases of* ASym_a.

Proof. Well, we have the correct number of polynomials (they are labelled by weak compositions). Next, with reverse lexicographic term order, the leading term of $\mathfrak{D}_a$ is x^a, so they are linearly independent. ♡

We remark that $\mathfrak{D}_a$ is non-symmetric MacDonald polynomial at $q = t = \infty$ (or 0). But the point is, we can get another formula for $\mathfrak{D}_a$.

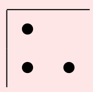
To do this, take a, we write it backword, denoted by $r(a)$, and take its Young diagram $D(r(a))$, and augment by adding a column 0 at the left, called the ***basement***.

Now consider fillings, called ***semi-skyline filling***, such that:

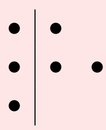
1. entries don't repeat in a column.
2. entries weakly decreasing left to right along rows.
3. every triple is inversion triple (for definition of inversion triple, see Definition 2.19).
4. basement box b_i is labelled by $n + 1 - i$.

Let $\mathfrak{D}\,\mathrm{SST}(a)$ be the set of key semi-skyline fillings for a.

Let $a = 021$, then we need to draw the Young diagram of 120:



Next, we need to add the basement, and we get (could not get the horizontal line appear after the first dot...)



Filling numbers in, we end up with the following

$$
\begin{array}{c|cc} 3 & 3 & \\ 2 & 2 & 2 \\ 1 & & \end{array}
\qquad
\begin{array}{c|cc} 3 & 3 & \\ 2 & 2 & 1 \\ 1 & & \end{array}
\qquad
\begin{array}{c|cc} 3 & 3 & \\ 2 & 1 & 1 \\ 1 & & \end{array}
\qquad
\begin{array}{c|cc} 3 & 2 & \\ 2 & 1 & 1 \\ 1 & & \end{array}
\qquad
\begin{array}{c|cc} 3 & 1 & \\ 2 & 2 & 2 \\ 1 & & \end{array}
$$

Theorem 7.8.

$$\mathfrak{D}_a = \sum_{T \in \mathfrak{D}\,\mathrm{SST}(a)} x^{\mathrm{wt}(T)}$$

with $\mathrm{wt}(T) = (a_1, \ldots, a_n)$ *with* a_i *equal the number of* i*'s not counting basement.*

We remark that $\{\mathfrak{D}_a\}$ does not have positive structure coefficients. Multiply $\mathfrak{D}_a$ are correspond to tensor products of some modules, and hence we should not expect the expansion is positive.

First, let's ask, what's $\{\mathfrak{D}_a\}|_{\mathrm{Sym}}$, i.e. restrict Demazure characters to symmetric polynomials.

Recall from assignment we have the Grassmannian permutations, in particular they are 2143-avoiding. Thus, we see for any Grassmannian permutation, we have $\mathfrak{S}_{\mathrm{Grass}} = s_\lambda = \mathfrak{D}_a$ where Grass is any Grassmannian permutation.

Theorem 7.9. *We have the following lift relations*

$$
\begin{tikzcd}
\mathfrak{S}_w & \mathfrak{D}_a \\
& s_\lambda \arrow[ul, hook] \arrow[u, hook]
\end{tikzcd}
$$

Before we think about the lift relation, let's ask what's the right $\mathfrak{D}_a$ for s_λ.

Suppose we have $s_\lambda(x_1, \ldots, x_n)$. Then we need $a = \lambda 0^k$ so that its of length n. Then $\mathfrak{S}_{r(a)} = s_\lambda(x_1, \ldots, x_n)$ where we recall $r(a)$ is write a in reverse order. But then we know $\mathfrak{S}_{r(a)} = \mathfrak{D}_{r(a)}$.

Example 7.10. Let $\lambda = (4, 2, 1)$, then $s_\lambda(x_1, \ldots, x_5) =$? (it is really long!). THen we have $\mathfrak{S}_{00124}(x) = \mathfrak{S}_{123693578}(x) = \mathfrak{D}_{00124}$.

In fact, if a is weakly increasing, then $\mathfrak{D}_a = s_{r(a)}(x_1, \ldots, x_{\ell(a)})$.

This tells us when is $\mathfrak{D}_a$ a symmetric polynomial. All the above still holds after you add a bar on top (i.e. all the above are true for K-theory version as well).

Next, we have the following refinement relations:

$$
\begin{tikzcd}
\mathfrak{S}_a \arrow[r] & \mathfrak{D}_a \\
& s_\lambda \arrow[ul, hook] \arrow[u, hook]
\end{tikzcd}
$$

The refinment is due to Shimozono-Yu in 2021. Now get back to the world of Sym and QSym, we have

$$\begin{tikzcd}
\mathfrak{S}_a \arrow[r] & \mathfrak{D}_a & & \mathfrak{F}_a & \\
& s_\lambda \arrow[ul, hook] \arrow[u, hook] \arrow[r] & S_\alpha \arrow[r] & F_\alpha \arrow[u, hook] \arrow[r] & M_\alpha
\end{tikzcd}$$

The natural question to ask is, do we have the following dash arrows and polynomials:

$$\begin{tikzcd}
\mathfrak{S}_a \arrow[r] & \mathfrak{D}_a \arrow[r, dashed] & ? \arrow[r, dashed] & \mathfrak{F}_a \arrow[r, dashed] & ? \\
& s_\lambda \arrow[ul, hook] \arrow[u, hook] \arrow[r] & S_\alpha \arrow[u, hook, dashed] \arrow[r] & F_\alpha \arrow[u, hook] \arrow[r] & M_\alpha \arrow[u, hook, dashed]
\end{tikzcd}$$

The good thing is, we indeed have all the arrows and polynomials:

$$\begin{tikzcd}
\mathfrak{S}_a \arrow[r] & \mathfrak{D}_a \arrow[r] & \mathfrak{Q}_a \arrow[r] & \mathfrak{F}_a \arrow[r] & \mathfrak{M}_a \\
& s_\lambda \arrow[ul, hook] \arrow[u, hook] \arrow[r] & S_\alpha \arrow[u, hook] \arrow[r] & F_\alpha \arrow[u, hook] \arrow[r] & M_\alpha \arrow[u, hook]
\end{tikzcd}$$

where $\mathfrak{Q}_a$ are the quasikey polynomials and $\mathfrak{M}_a$ are the monomial slide polynomials.

The K-theory arrows are almost true, except we miss the Shimozono-Yu arrow. In particular, $\overline{\mathfrak{Q}}_a$ are called quasi-Lascoux polynomials and $\overline{\mathfrak{M}}_a$ are called the monomial glide polynomials.

We also have the following refinment, where $\mathfrak{A}_a$ are the Demazure atoms:

$$\begin{tikzcd}
& & \mathfrak{A}_a & & \\
\mathfrak{D}_a \arrow[r] & \mathfrak{Q}_a \arrow[ur] \arrow[dr] & & & \\
& & \mathfrak{F}_a \arrow[dr] & & \mathfrak{X}_a \\
& & & \mathfrak{M}_a \arrow[ur] &
\end{tikzcd}$$

The funny thing here is that, the change of basis matrix for $\mathfrak{Q}_a$ to $\mathfrak{A}_a$ is exactly the same matrix for $\mathfrak{Q}_a$ to $\mathfrak{X}_a$. Denote this change of basis matrix by u, we might expect if we hit u with $\mathfrak{F}_a$, we might get something. This is indeed the case, and they are called the pions. Thus, we get the following, where the letters on the arrow would denote change of basis matrix:

$$\begin{tikzcd}
 & & \mathfrak{A}_a \arrow[dr, "u^{-1}Au"] & & \\
\mathfrak{D}_a \arrow[r] & \mathfrak{Q}_a \arrow[ur, "u"] \arrow[dr, "A"'] & & \mathfrak{P}_a \arrow[dr] & \\
 & & \mathfrak{F}_a \arrow[ur, "u"'] \arrow[dr] & & \mathfrak{X}_a \\
 & & & \mathfrak{M}_a \arrow[ur] &
\end{tikzcd}$$

The amazing thing is that we have arrows from $\mathfrak{A}_a$ to $\mathfrak{P}_a$ and $\mathfrak{P}_a$ to $\mathfrak{X}_a$, as even we have the matrices expansion, we should not expect the entries to be positive (remember we have arrows iff there is a refinment relation), but they do, in both cases.

Recall last class we had the following picture

$$\begin{tikzcd}
 & & & \mathfrak{A} \arrow[r] & \mathfrak{P} \arrow[r] & \mathfrak{X}_a \\
\mathfrak{S}_a \arrow[r] & \mathfrak{D}_a \arrow[r] & \mathfrak{Q}_a \arrow[r] \arrow[ur] & \mathfrak{F}_a \arrow[r] \arrow[ur] & \mathfrak{F}_a \arrow[ur] & \\
 & s_\lambda \arrow[r] \arrow[u, hook] \arrow[ul, hook] & S_\alpha \arrow[r] \arrow[u, hook] & F_\alpha \arrow[r] \arrow[u, hook] & M_\alpha \arrow[u, hook] &
\end{tikzcd}$$

We seen

$$\mathfrak{A}_a = \sum_{T \in \mathfrak{A}\,\mathrm{SST}(a)} x^{\mathrm{wt}(T)}$$

where we recall $\mathfrak{A}\,\mathrm{SST}$ is the set of semistandard composition tableaux of shape a such that:

1. entries don't repeat in columns
2. rows weakly decreasing
3. every triple is inversion
4. the first column entries match row indices.

The K-theory version of the Demazure atoms are called Lascoux atoms. In this case, we have

$$\overline{\mathfrak{A}}_a = \sum_{T \in \overline{\mathfrak{A}}\,\mathrm{SST}} \beta^{\mathrm{ex}(T)} x^{\mathrm{wt}(T)}$$

and we need to figure out what the four conditions above are, for set-valued $\overline{\mathfrak{A}}\,\mathrm{SST}(a)$. They are given by

1. entries don't repeat in columns, as sets
2. rows weakly decreasing setwise
3. every triple of anchors is inversion, where ***anchor*** is the biggest thing in its box, and we call an number is ***free*** if its not an anchor

4. first column anchor match row indices
5. every free entry is with the smallest anchor in its column subject to (2).

Example 7.11. Lets look at an example, say $\overline{\mathfrak{A}}_{1032}$. Then we can do

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline 3,2 & 2 & 2 \\ \hline 4 & 4,3,1 & \\ \hline \end{array}$$

The 5th condition applies to the 2's in the first column, which tells the 2 cannot live with the 1 in the first row, because it will not be free anymore, and so on.

Proposition 7.12. *$\{\mathfrak{A}_a\}$ is a basis of* ASym_n.

Proof. With respect to a reverse lexicographic order, the leading term fo $\mathfrak{A}_a$ is x^a. ♡

Proposition 7.13. *Demazure atoms refines quasiSchur*

$$S_\alpha(x_1, ..., x_n) = \sum_{a^+=\alpha} \mathfrak{A}_a$$

Proof. By definition. ♡

There is an alternative definition of Demazure atoms. Recall π_i operator, and we can define $\hat{\pi} = \pi_i - 1$. This has a K-theory version which is $\hat{\overline{\pi}}_i = \overline{\pi} - 1$.

Now recall $w(a)$ is the shortest permutation with $aw(a) = \mathrm{sort}(a)$. Then we can define

$$\mathfrak{A}_a := \hat{\pi}_{w(a)} x^{\mathrm{sort}(a)}$$

and

$$\overline{\mathfrak{A}}_a := \hat{\overline{\pi}}_{w(a)} x^{\mathrm{sort}\, a}$$

The first equivalence of definitions is well-known, but the second equivalence of definitions is pretty new, due to Buciumas, Scrimshaw, Weber in 2020.

By looking at the operators, we get the following.

Proposition 7.14. *Demazure atoms refine Demazure characters, i.e.*

$$\mathfrak{D}_a = \sum_{\substack{\mathrm{sort}(b)=\mathrm{sort}(a) \\ w(b) \leq w(a)}} \mathfrak{A}_b$$

where the second inequality is strong Bruhat order. The K-theory version also holds.

Corollary 7.14.1.

$$s_\lambda = \sum_{a \in S_\lambda} \mathfrak{A}_a$$

where S_λ is the set of permutations on λ. The K-theory version also holds.

We note the Demazure characters don't have positive structure coefficients. The Demazure atoms also don't have positive structure coefficients.

However, we have the following conjecture.

Conjecture 7.15 (Reiner-Shimozono)**.** $\mathfrak{D}_a \cdot \mathfrak{D}_b$ is a positive sum of $\mathfrak{A}_c$.

There is also the K-theory version conjecture, which is due to Monical-Pechenik-Searles in 2021. This is the end of Demazure atoms, and we are moving to key polynomials $\mathfrak{Q}_a$.

Definition 7.16 (Assaf-Searles, 2018)**.** We define the ***quasikey polynomial*** as

$$\mathfrak{Q}_a = \sum_{\substack{b^+=a^+ \\ b \geq a}} \sum_{\mathfrak{A}\mathrm{SST}(b)} x^{\mathrm{wt}(T)} = \sum_{\substack{b^+=a^+ \\ b \geq a}} \mathfrak{A}_b$$

where $b \geq a$ means b dominance a. We also define the K-theory version, called ***quasiLascoux polynomial***, as

$$\overline{\mathfrak{Q}}_a = \sum_{\substack{b^+=a^+ \\ b \geq a}} \sum_{T \in \overline{\mathfrak{A}}\,\mathrm{SST}} x^{\mathrm{wt}(T)} = \sum_{\substack{b^+=a^+ \\ b \geq a}} \overline{\mathfrak{A}}_b$$

The K-theory definition is due to Monical-Pechenik-Searles in 2021.

Example 7.17. We have

$$\mathfrak{Q}_{103} = \mathfrak{A}_{103} + \mathfrak{A}_{130}$$

Note we are missing the $\mathfrak{A}_{013}$ piece, and if we add this, we get back to $\mathfrak{S}_\alpha$.

We note $\mathfrak{Q}_a \to \mathfrak{A}_a$ is built-in in the definition.

Theorem 7.18 (Assaf-Searles)**.** *We have the following expansion*

$$\mathfrak{Q}_a \to \mathfrak{F}_a$$

given by

$$\mathfrak{Q}_a = \sum_T \mathfrak{F}_{\mathrm{wt}(T)}$$

where the sum is over $T \in \bigcup \mathfrak{A}\,\mathrm{SST}(b)$ with $b^+ = a^+, b \geq a$, with T being quasi-Yamanouchi and T containing the support of a.

Example 7.19. We consider $\mathfrak{Q}_{103}$. The tableaux are given by

$$\begin{array}{ccccc} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ & & & & \\ \boxed{3}\boxed{3}\boxed{3} & \boxed{3}\boxed{3}\boxed{2} & \boxed{3}\boxed{3}\boxed{1} & \boxed{3}\boxed{2}\boxed{2} & \boxed{3}\boxed{2}\boxed{1} \end{array}$$

$$\begin{array}{lll} \boxed{1} & & \\ \boxed{2} & \boxed{2} & \boxed{2} \end{array} \qquad \begin{array}{lll} \boxed{1} & & \\ \boxed{2} & \boxed{2} & \boxed{1} \end{array}$$

Then we to check the quasi-Yamanouchi condition. This gives

$$\begin{array}{lll} \boxed{1} & & \\ & & \\ \boxed{3} & \boxed{3} & \boxed{3} \end{array} \qquad \begin{array}{lll} \boxed{1} & & \\ & & \\ \boxed{3} & \boxed{3} & \boxed{1} \end{array} \qquad \begin{array}{lll} \boxed{1} & & \\ \boxed{2} & \boxed{2} & \boxed{2} \end{array} \qquad \begin{array}{lll} \boxed{1} & & \\ \boxed{2} & \boxed{2} & \boxed{1} \end{array}$$

Next, we need to check T containing the support of a. This rule out two more and we are left with

$$\begin{array}{lll} \boxed{1} & & \\ & & \\ \boxed{3} & \boxed{3} & \boxed{3} \end{array} \qquad \begin{array}{lll} \boxed{1} & & \\ & & \\ \boxed{3} & \boxed{3} & \boxed{1} \end{array}$$

Hence we conclude

$$\mathfrak{Q}_{103} = \mathfrak{F}_{103} + \mathfrak{F}_{202}$$

We see this is indeed the case.

We are going to finish quasikey and start pion and kion.

Proposition 7.20. *We have* $\mathfrak{Q}_a \in \mathrm{QSym}_n$ *if and only if* $a = 0^k\alpha$. *In this case,* $\mathfrak{Q}_{0^k\alpha} = S_\alpha(x_1, \ldots, x_n)$.

The last topic on quasikey is to expand key in terms of quasikey.

Definition 7.21. A ***left swap*** of a exchanges entries $a_i < a_j$ with $i < j$. We use $\mathrm{lswap}(a)$ be the set of all b obtainable by a sequence of left swaps.

Example 7.22. If $a = 103$, then using one left swap we get $\{130, 301\}$, use another left swap we get 310, and use the empty left swap we get 103. Hence

$$\mathrm{lswap}(103) = \{130, 301, 310, 103\}$$

Remark 7.23. We see $\mathrm{lswap}(a)$ is exactly the set of b with $\mathrm{sort}(a) = \mathrm{sort}(b)$ and $w(b) \le w(a)$.

Corollary 7.23.1.

$$\mathfrak{D}_a = \sum_{b \in \mathrm{lswap}(a)} \mathfrak{A}_b$$

Definition 7.24. Let $Q\,\text{lswap}(a)$ be the set of $b \in \text{lswap}(a)$ such that for all $c \in \text{lswap}(a)$ with $c^+ = b^+$, we have $c \geq b$.

Example 7.25. Recall $\text{lswap}(103) = \{130, 301, 310, 103\}$, then we see $Q\,\text{lswap}(103) = \{301, 103\}$ by a simple observation.

Proposition 7.26.

$$\mathfrak{D}_a = \sum_{b \in Q\,\text{lswap}(a)} \mathfrak{Q}_b$$

Proof. We see $\mathfrak{D}_a = \sum_{b \in \text{lswap}(a)} \mathfrak{A}_b$, but for $b \in Q\,\text{lswap}(a)$ we see $\sum_{\substack{c \in \text{lswap}(a) \\ c^+ = b^+}} \mathfrak{A}_c = \mathfrak{Q}_b$. ♡

Example 7.27. Continue the above example, we see this means

$$\mathfrak{D}_{103} = \mathfrak{Q}_{103} + \mathfrak{Q}_{301}$$

Conjecture 7.28 (Monical-Pechenik-Searles,2021)**.** If $\overline{\mathfrak{Q}_a} = \sum_b M_b^a(\beta)\overline{\mathfrak{F}_b}$, then $\sum_b M_b^a(-1) \in \{0, 1\}$.

Equivalently, we see the above is the same as $\overline{\mathfrak{Q}}_a(x = 1, \beta = -1) \in \{1, 0\}$, as we know $\overline{\mathfrak{F}_b}(x = 1, \beta = -1) = 1$, which is due to Smirnov-Tutubalina, 2021.

Example 7.29. Let $a = 0662$, we see

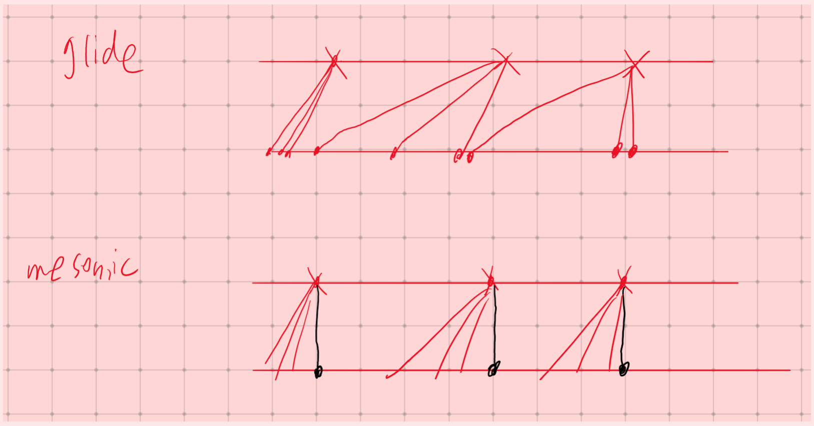
$$\sum_b M_b^a(\beta) = 36 + 94\beta + 75\beta^2 + 16\beta^3 = 1$$

8 Pion& Kion

Now we switch topic, first recall the definition of glide in Definition 4.47.

Definition 8.1. A glide of a is ***mesonic*** if it is obtainable without applying $0p \to p0$ to initial non-zero positions.

To picture this, we know general glide smear things to the left, and mesonic glide can only smear the numbers in a more rigid way:



Example 8.2. Let $a = 03002$, then $b = 31002$ is not a mesonic glide. On the other hand, $b' = 21102$ is mesonic.

Definition 8.3. The ***kaon*** is defined by

$$\overline{\mathfrak{P}}_a := \sum_{b \text{ mesonic glide of } a} \beta^{\#\text{red}} x^b$$

Definition 8.4. The ***pion*** is defined by

$$\mathfrak{P}_a = \overline{\mathfrak{P}}_a(\beta = 0) = \sum_{b \text{ mesonic slide of } a} x^b$$

Now recall the refinment relations

$$\begin{tikzcd}
\mathfrak{Q} \arrow[r] \arrow[d] & \mathfrak{A} \arrow[d] \\
\mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{P} \arrow[d] \\
\mathfrak{M} \arrow[r] & \mathfrak{X}
\end{tikzcd}$$

We already know the top arrow, which is given by $\overline{\mathfrak{Q}}_a = \sum_{b^+=a^+, b\geq a} \overline{\mathfrak{A}}_a$. This also gives $\overline{\mathfrak{P}}$ formula.

Theorem 8.5.

$$\overline{\mathfrak{F}}_a = \sum_{\substack{b^+=a^+\\ b\geq a}} \overline{\mathfrak{P}}_a$$

Proof. Let g be glide of a. Suppose a has non-zero entries in positions $n_1 < \ldots < n_\ell$. By summing initial segments of g, we know which entries of g came from a.

Let i_j be the rightmost position where g has a contribution from position n_j of a. Then we can obtain g from a by using $0p \to p0$ to move each n_j entry to i_j. Call this weak composition b. Then we have $b^+ = a^+$, and $b \geq a$ since we slide things to the left. The point is, we can get g out of b, and we already done all the $0p \to p0$ that would cross zero entries when doing a to b, we don't have to do $0p \to p0$ moves at i_j. Hence every glide of a is going to be mesonic glide of a unique b, where $b^+ = a^+$ and $b \geq a$.

Thus the proof follows. ♡

Corollary 8.5.1. $\{\overline{\mathfrak{P}}_b\}$ *is a basis of* $\mathrm{ASym}_n[\beta]$.

Proof. The matrix describing $\overline{\mathfrak{F}_a} \to \overline{\mathfrak{P}}_b$ is triangular with 1 on the diagonal. So it is invertible over $\mathbb{Z}$ and we know $\{\overline{\mathfrak{F}}_a\}$ is a basis. ♡

Kaons don't have positive structure coefficients.

Conjecture 8.6 (Monical-Pechenik-Searles, 2021)**.** $\overline{\mathfrak{P}}_a \cdot \overline{\mathfrak{F}}_b$ is $\overline{\mathfrak{P}}$-positive.

The question is also open when set $\beta = 0$.

Conjecture 8.7. $\overline{\mathfrak{P}}_a \cdot \overline{s}_\lambda$ is $\overline{\mathfrak{P}}$-positive.

Theorem 8.8 (Searles, 2020)**.** $\mathfrak{P}_a \cdot s_\lambda$ *is* $\mathfrak{P}$*-positive.*

The next arrow we are going to talk about is $\mathfrak{A} \to \mathfrak{P}$.

Definition 8.9. Let $T \in \overline{\mathfrak{A}}\mathrm{SST}(a)$, for $i \in T$, let $i^\uparrow$ be the next largest label in T. Say T is ***meson-highest*** if for each i, the leftmost i is either:

1. an anchor in first column
2. there is an $i^\uparrow$ weakly to its right in a different box.

We see this is similar to the quasi-Yamanouchi condition, if we flattening down the alphabet.

Theorem 8.10 (Monical-Pechenik-Searles, 2021)**.**

$$\overline{\mathfrak{A}}_a = \sum_{\substack{meson-highest \\ T \in \overline{\mathfrak{A}}\mathrm{SST}(a)}} \beta^{\#\,\mathrm{ex}(T)} \overline{\mathfrak{P}}_{\mathrm{wt}(T)}$$

Example 8.11. We compute $\overline{\mathfrak{A}}\mathrm{SST}(102)$, as any larger examples would blow-up to a huge number of data. We have

$$\begin{array}{l} \boxed{1} \\ \\ \boxed{3}\boxed{3} \end{array} \quad \begin{array}{l} \boxed{1} \\ \\ \boxed{3}\boxed{3}\boxed{2} \end{array} \quad \begin{array}{l} \boxed{1} \\ \\ \boxed{3}\boxed{2} \end{array} \quad \begin{array}{l} \boxed{1} \\ \\ \boxed{32}\boxed{2} \end{array} \quad \begin{array}{l} \boxed{1} \\ \\ \boxed{3}\boxed{31} \end{array}$$

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 3 & 321 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 3 & 21 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 32 & 31 \\ \hline \end{array}$$

Now we look at the meson-highest tableaux, which are

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 3 & 31 \\ \hline \end{array}$$

This suggests

$$\overline{\mathfrak{A}}_a = \overline{\mathfrak{P}}_{103} + \beta\overline{\mathfrak{P}}_{202}$$

One can check this is indeed the case, by translate the mesonic glides to the above two tableaux indeed gives all the elements in $\overline{\mathfrak{A}}\,\mathrm{SST}(102)$.

Next time, we do $\mathfrak{F} \to \mathfrak{M}$, where we need to show it expand positively, which are called monomial slide/glide polynomials. Also, those $\mathfrak{M}$ have positive structure coefficients.

Definition 8.12. We define the ***monomial slide polynomial***

$$\mathfrak{M}_a = \sum_{\substack{b^+ = a^+ \\ b \geq a}} \mathfrak{X}_b$$

Example 8.13. We have

$$\mathfrak{M}_{0203} = x^{0203} + x^{2003} + x^{0230} + x^{2030} + x^{2300}$$

Recall we have three local moves on weak kompositions:

1. $0p \to p0$.
2. $0p \to qr$ with $q + r = p$ and $r, q > 0$.
3. $0p \to qs$ with $q + s = p + 1$ and $q, s > 0$.

For $S \subseteq 1, 2, 3$, write $ш_S(a)$ for the set of weak kompositions obtainable from a by rules in S, e.g if $S = \{1, 2, 3\}$ then we can use rule 1 to 3, and if $S = \{1, 2\}$ then we can only use rule 1 and 2 but not 3. Then

$$\overline{\mathfrak{F}}_a = \sum_{b \in ш_{123}(a)} \beta^{\#\mathrm{red}} x^b$$

$$\mathfrak{F}_a = \sum_{b \in ш_{12}(a)} x^b$$

On the other hand, we also know

$$\mathfrak{M}_a = \sum_{b \in ш_1(a)} x^b$$

Thus, the natural guess of the definition of the K-theory version of $\mathfrak{M}$ should be the following:

$$\overline{\mathfrak{M}}_a = \sum_{b \in \text{Ш}_{13}(a)} \beta^{\#\text{red}} x^b$$

This is a conjectural definition[2].

From this perspective, we also have the silly

$$\mathfrak{X}_a = \sum_{b \in \text{Ш}_{\varnothing}(a)} x^b$$

and hence the K-ified polynomial should be

$$\overline{\mathfrak{X}}_a = \sum_{b \in \text{Ш}_3(a)} \beta^{\#\text{red}} x^b$$

This is conjectural as wel.

Conjecture 8.14 (Lam-Pylyavskyy)**.** $\{\overline{\mathfrak{M}}_a\} \cap \mathrm{QSym} = \{\overline{M}_a\}$

Theorem 8.15. $\{\mathfrak{M}_a\}$ *is a basis of* ASym.

Proof. With respect to a linear extension of dominance order, the matrix expanding $\mathfrak{M}_a$ into $\mathfrak{X}_b$ is uni-triangular. Thus it is invertible over $\mathbb{Z}$. ♡

Theorem 8.16. $\mathfrak{M}_a \in \mathrm{QSym}_n$ *if and only if* $a = 0^k\alpha$*. In this case,* $\mathfrak{M}_a = M_\alpha$*.*

Proof. Suppose a is not quasistrong. Then there exists $i < k$ with $a_{i-1} > 0$, $a_i = 0$, $a_h > 0$. Then $x^a = x_1^{a_1} \dots x_{i-1}^{a_{i-1}} x_i^0 \dots x_k^{a_k}$ is a term of $\mathfrak{M}_a$. But $x_1^{a_1} \dots x_{i-1}^{a_{i-1}} \dots x_k^{a_k}$ is not a term and so $\mathfrak{M}_a$ is not in QSym as desired.

Suppose a is quasistrong, i.e. $a = 0^k\alpha$, then the b with $b^+ = \alpha$ are exactly the b with $b^+ = a^+$ and $b \geq a$. Thus we see $\mathfrak{M}_a = M_\alpha$ as desired. ♡

Thus, we concluded another arrow in the diagram, and we have two things left to do. First, we want to look at the arrow $\mathfrak{F}_a \to \mathfrak{M}_a$, and we want to talk about the positivity of structure coefficient of $\mathfrak{M}_a$.

Before talk about the arrow, we need a new definition.

Definition 8.17. We say $b \trianglerighteq a$ if:

1. $b \geq a$
2. if $c \geq a$ and $c^+ = b^+$ then $c \geq b$.

Proposition 8.18. $\mathfrak{F}_a \to \mathfrak{M}_b$ *with*

$$\mathfrak{F}_a = \sum_{\substack{b \trianglerighteq a \\ b^+ \vDash a^+}} \mathfrak{M}_b$$

[2]In the sense that it should be correct, but our prof don't have a proof yet

Proof. By the definition. ♡

Example 8.19. Consider $\mathfrak{F}_{0203}$. We see

$$\begin{aligned}\mathfrak{F}_{0203} = \mathfrak{M}_{0203} + \mathfrak{M}_{0212} + \mathfrak{M}_{0221} + \mathfrak{M}_{1103}\\ + \mathfrak{M}_{1112} + \mathfrak{M}_{1121} + \mathfrak{M}_{2111}\end{aligned}$$

Definition 8.20. For weak compositions a, b, let $S(a, b)$ be the set of all pairs (a', b') such that:

1. $(a')^+ = a^+$ and $(b')^+ = b^+$
2. $a' \geq a$ and $b' \geq b$
3. $a' + b'$ is a strong composition.

Definition 8.21. Let a, b be weak compositions. Fix $(a', b') \in S(a, b)$, let c be a weak composition with $c^+ = a' + b'$. Let c_a be the weak composition with 0's in the same position as c, but the other entries are a'_i instead of $(a' + b')_i$. So, in particular, $c_a^+ = (a')^+ = a^+$. Define the same for c_b, i.e. c_b have 0's in the same position as c but other entries are b'_i instead of $(a' + b')_i$. In particular, note $c_a + c_b = c$.

Now, for every pair $(a', b') \in S(a, b)$, let $\mathrm{Bump}(a', b')$ be the dominance-smallest (shove to the right as possible) weak composition such that:

1. $\mathrm{Bump}(a', b')^+ = a' + b'$
2. $\mathrm{Bump}(a', b')_a \geq a$
3. $\mathrm{Bump}(a', b')_b \geq b$

Definition 8.22. The ***overlapping slide product*** of a and b is the formal sum $a ш_o b$ of all the weak compositions $\mathrm{Bump}(a', b')$ for $(a', b') \in S(a, b)$.

Example 8.23. Let $a = 0102, b = 1001$, then $S(a, b)$ is the set

$$\{(0102, 1010), (0120, 1001), (012, 101), (012, 110), (120, 101), (102, 110), (12, 11)\}$$

Then

$$a \underset{o}{ш} b = 1112 + 1121 + 11\underline{0}3 + 2\underline{0}12 + 12\underline{0}2 + 2\underline{0}21 + 2\underline{00}3$$

where in the above, we use underline to denote the 0 added during Bump.

Theorem 8.24. *The $\{\mathfrak{M}_a\}$ basis has positive structure coefficients. In particular,*

$$\mathfrak{M}_a \cdot \mathfrak{M}_b = \sum c_{ab}^c \mathfrak{M}_c$$

where c_{ab}^c is the multiplicity of c in $a ш_o b$

Proof. By definition, $\mathfrak{M}_a \mathfrak{M}_b = \sum_{(a',b')} x^{a'+b'}$ with sum over (a', b') such that $(a')^+ = a$ and $(b')^+ = b$. Our rule collects together such monomials as monomial slide polynomials. ♡

Example 8.25. We see by the above computation, we get

$$\mathfrak{M}_{0102} \cdot \mathfrak{M}_{1001} = \mathfrak{M}_{1112} + \mathfrak{M}_{1121} + \mathfrak{M}_{1103} + \mathfrak{M}_{1202} + \mathfrak{M}_{2012} + \mathfrak{M}_{2021} + \mathfrak{M}_{2003}$$

Theorem 8.26. *For all* 8 *bases of* ASym*, the product of a basis element times a Schur polynomial expands positivity in the same basis.*

For $\mathfrak{S}_a$, this above theorem is from geometry (i.e. $\mathfrak{S}_a \cdot s_\lambda$ is just product of two Schubert polynomial, hence positive as desired).

For $\mathfrak{F}_a$ and $\mathfrak{M}_a$ and $\mathfrak{X}_a$, this follows from each having positive structure coefficients and $s_\lambda \to \mathfrak{F}_a \to \mathfrak{M}_a \to \mathfrak{X}_a$.

For $\mathfrak{D}_a$ and $\mathfrak{A}_a$, see Haglund, Luoto, Mason and van Willigenburg in 2011.

For $\mathfrak{Q}_a$ and $\mathfrak{P}_a$, see Searles 2020.

Conjecture 8.27. The same is true for K-version of the 8 bases and symmetric Grothendieck polynomials.

This is about the end of the general ASym polynomials. Next we are going to focous on Schubert polynomials. But before this, let's ask:

1. What are all the polynomials telling us?
2. what do these weird bases mean?
3. Three of those 8 polynomials are coming from MacDonald theory, what about the rest 5?

9 Schubert Polynomials, Again

The rest of the term will be about Schubert polynomials.

So, Schubert polynomials, and we will think about the space M_n of $n \times n$ matrices, which is just $\mathbb{C}^{n^2}$.

Inside the space of matrices, what happen when I multiply them together? Say, we want to look at all $A \in M_n$ such that $A^2 = 0$, denote this set as N. This is an algebraic variety because $A \in N$ is determined by vanishing of polynomials. For example, say we have

$$A = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

then we have 9 things to compute, and we need to set all of them equal 0. In general, we need to set n^2 polynomials to 0.

The next question is, are these only polynomial relations satisfied on N? Of course not, as we recall $N = V(\mathcal{I}) = \operatorname{Spec} \mathbb{C}[x^1, ..., x^{n^2}]/\sqrt{\mathcal{I}}$, hence we have the entire ideal $\mathcal{I}$ generated by the n^2 polynomials vanishes on N.

This is easy to see, and the next question is, are we done yet? Are there anything else? Is there any secret equations? Well, we are missing the trace equation. Since those polynomials are nilpotent, we know its trace is always zero, and this is a linear term.

Beside the trace, is there anything else? Nope, and this is the following theorem.

Theorem 9.1. *Let $\mathcal{I}(N)$ be the set of polynomials that vanish identically on N, then $\mathcal{I}(N)$ is generated by the n^2 terms and the trace.*

The moral of the story is, the generator of $\mathcal{I}$ is not always the full generators of $\sqrt{\mathcal{I}}$.

Definition 9.2. For I a set of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$, we define

$$V(I) = V(\mathcal{I}) = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : \forall f \in \mathcal{I}, f(x) = 0\}$$

where $\mathcal{I}$ is the ideal generated by the set of polynomials I.

Definition 9.3. For V a set of points in $\mathbb{C}^n$, we define

$$I(V) = \{f \in \mathbb{C}[x_1, \ldots, x_n] : \forall x \in V, f(x) = 0\}$$

The above two definitions almost look like inverse of each other, and the question of secret equations above is just the question of how much is V the inverse of I. In other word, when does $I(V(\mathcal{I})) = \mathcal{I}$?

Observe if $f^k \in I(V(\mathcal{I}))$, then we see $\forall p \in V(\mathcal{I})$ we have $f^k(p) = 0$. But this is $(f(p))^k = 0$ and hence we must have $f(p) = 0$ as $\mathbb{C}$ is a field. Thus we see $f \in I(V(\mathcal{I}))$.

Definition 9.4. For $\mathcal{I}$ an ideal, we define the ***radical*** of I as

$$\sqrt{\mathcal{I}} = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f^k \in \mathcal{I} \text{ for some } k > 0\}$$

We say $\mathcal{I}$ is ***radical*** if $\mathcal{I} = \sqrt{\mathcal{I}}$.

Proposition 9.5. $\sqrt{\sqrt{\mathcal{I}}} = \sqrt{\mathcal{I}}$ *for all ideal $\mathcal{I}$.*

Theorem 9.6 (Nullstellensatz)**.** *For any ideal $\mathcal{I}$ of $\mathbb{C}[x_1, \ldots, x_n]$, we have*

$$I(V(\mathcal{I})) = \sqrt{\mathcal{I}}$$

Thus, apply Nullstellensatz to $\mathcal{I}(N)$, we see this means some power k of trace sits inside the n^2 quadratic equations. To show we don't have any more secret equations, we just need to show $\mathcal{I}(N) = \sqrt{\mathcal{I}(N)}$. This is doable, but not always computationally easy.

This is the sort of geometry we want to think about, and the next topic is classical determinantal varieties.

Let M_n be the space of $n \times n$ matrices, consider the locus of singular matrices in M_n. It is a variety given by $V(\det)$, in this case, it is a hypersurface.

More generally, let M_{mn} be the space of $m \times n$ matrices, and this time we cannot talk about invertibility, but instead the rank. In particular, we can consider the locus of matrices of rank $\leq k$.

It is an algebraic variety, denoted by M^k_{mn}, cutout by all $(k+1) \times (k+1)$ minors (if you have a $k+1$ by $k+1$ minor refuse to die, then your rank is at least $k+1$).

Example 9.7. Consider M^1_{34}, so we have the following

$$\begin{bmatrix} x_1 & \dots & x_4 \\ y_1 & \dots & y_4 \\ z_1 & \dots & z_4 \end{bmatrix}$$

Then $M^1_{34} = V(\langle 2 \times 2 \text{ minors} \rangle)$, which is generated by, for example, $x_1 z_3 - x_3 z_1$ and so on.

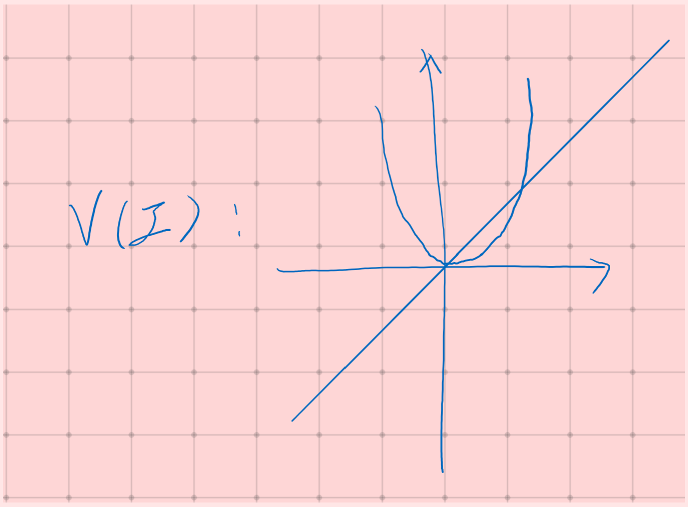
Thus, is there any secret equations this time? The answer is no this time, but it made people busy for a centuries or so.

Definition 9.8. An ideal $\mathcal{I}$ is prime if $fg \in \mathcal{I}$ implies $f \in \mathcal{I}$ or $g \in \mathcal{I}$.

Lemma 9.9. *$\mathcal{I}$ is prime implies $\mathcal{I}$ is radical.*

Proof. Say $f^k \in \mathcal{I}$, then $f^{k-1} f \in \mathcal{I}$ and so either $f \in \mathcal{I}$ or $f^{k-1} \in \mathcal{I}$, now induction follows. ♡

Example 9.10. Consider $\mathcal{I} := \langle (y - x^2)(y - x) \rangle$, then $V(\mathcal{I})$ is given by



Clearly this is two "graphs" glued together, and those two graphs are given by

$V(y-x)$ and $V(y-x^2)$. Its not hard to see $(y-x)$ and $(y-x^2)$ are primes, and hence $\mathcal{I}$ is "decomposed" into two prime ideals.

Theorem 9.11. *$\langle \mathrm{minor}_{(k+1)} \rangle$ is a prime ideal (hence radical), where $\mathrm{minor}_{(k+1)}$ is the set of $k+1$ by $k+1$ minors.*

The right context for those questions is "Schubert determinant ideals".

For $w \in S_n$, recall the rank matrix (see Definition 3.3). For example, if $w = 2143$, then the rank matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

We note this is indeed the "rank" of a matrix. To be precise, the (i,j) entry of the above matrix, is the rank of a submatrix in the permutation matrix associated with w.

Definition 9.12. The ***matrix Schubert variety*** X_w is the locus in M_n such that the northwest submatrix has rank at most r_{ij}, where r_{ij} is the (i,j)-entry of w's rank matrix.

Definition 9.13. The ***Schubert determinantal ideal*** is the ideal

$$\mathcal{I}_w = \langle \mathrm{minor}_{r_{ij}+1} \text{ in the northwest submatrix } Z_{ij} \rangle$$

So why we can put Schubert in front of everything? If we take X_w, and consider $X_w \cap \mathrm{GL}_n(\mathbb{C})$, and take a quotient by Borel subgroup B. This gives $X_w \cap \mathrm{GL}_n(\mathbb{C})/B \subseteq$ Flag, then we get the Schubert variety $\Omega_{w_0 \cdot w^{-1}}$.

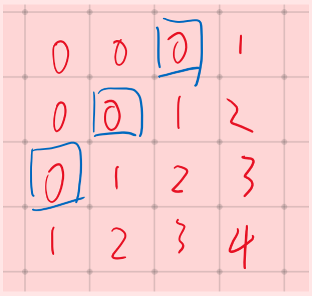
We want to understand what's going on with those, and the first question to ask is, is $\mathcal{I}_w$ radical, and if this is the case, is $\mathcal{I}_w$ is prime?

To do this, we will deform our space to something simpler.

Example 9.14. Take $w = 4321$, then the rank matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Most of the conditions in the Schubert determinantal ideal is redundant. In particular, all we need is the information from the blue boxes in the following:



To see this, note if we know, for example, the rank of the submatrix at $(3,1)$ is zero, then the rank of submatrices on top of this blue box must all be 0. If we know the submatrix at $(3,1)$ and $(2,2)$, then we know the submatrix at $(3,2)$ is at most 1, and so on.

Theorem 9.15 (Fulton)**.** *$\mathcal{I}_w$ is generated by the minors from the essential rank conditions.*

This is plausible because rank functions restricted to essential set determines the permutation. Apparently, people just call this the Fulton generators.

Let $R = \mathbb{C}[z_1, \ldots, z_n]$ be a polynomial ring and $\mathcal{I}$ an ideal of R. Then we have the quotient ring $R/\mathcal{I}$. If $\mathcal{I}$ is homogenous, then $R/\mathcal{I}$ is graded, and each graded piece is a $\mathbb{C}$-vector space.

Thus, the natural question is, what is $\dim_{\mathbb{C}}(R/\mathcal{I})^{(m)}$? Well, since this is a combinatorics course, of course we want to build the generating function:

$$H(t) := \sum_{m=0}^{\infty} \dim_{\mathbb{C}}(R/\mathcal{I})^{(m)} t^m$$

This is the Hilbert series of $R/\mathcal{I}$.

Example 9.16. Consider $\mathbb{C}[x,y,z]/(0)$, then

$$H(t) = 1 + 3t + 6t^2 + \binom{3+2}{2}t^3 + \binom{4+2}{2}t^4 + \ldots + \binom{m+2}{2}t^m + \ldots = \frac{1}{(1-t)^3}$$

Similarly, we see $\mathbb{C}[z_1, \ldots, z_n]/(0)$ gives $H(t) = \frac{1}{(1-t)^n}$.

Next, take $\mathcal{I} = (xy)$ and consider $\mathbb{C}[x,y,z]/(xy)$. In this case,

$$H(t) = 1 + 3t + 5t^2 + 7t^3 + (15-6)t^4 + \ldots + \left(\binom{m+2}{2} - \binom{m+2-2}{2}\right)t^m + \ldots = \frac{1-t^2}{(1-t)^3}$$

We observe in all three examples, the $H(t)$ always has the rational form, and it is of the form $H(t) = \frac{k(t)}{(1-t)^n}$, if we are working with n variables, where the $k(t)$ is called the ***k-polynomial of*** $R/\mathcal{I}$. There is another piece of information we can obtain from this.

Example 9.17. Take $\mathcal{I} = (xy)$ and $R = \mathbb{C}[x,y,z]$. In this case, we have $k(t) = 1 - t^2$. Next, we make a change of variable $t \mapsto 1 - t$, and then $k(t)$ becomes $2t - t^2$, and we call the lowest degree part to be the ***degree*** of $R/\mathcal{I}$. In this case, the degree is $2t$. If the setting is nice, then the coefficient of the degree gives the number of irreducible components of $R/\mathcal{I}$, and t tells you the codimension of those irreducible components are 1.

We also note normal people would say the degree is 2, not $2t$.

Next, we want to milk the Schubert polynomials out of this. Thus, let's go back to the setup with a matrix of variables $(z_{ij})_{1\le i,j\le n}$. Next, we say z_{ij} has multidegree $\hat{e}_i = (0, \dots, 1, \dots, 0)$ where we get 1 at ith position. For example, in this case, the polynomial $z_{13}z_{25}z_{14}z_{23}^2z_{78}$ has multidegree

$$(2,3,0,0,0,0,1,0)$$

Now consider $R(z_{ij})/\mathcal{I}$ with $\mathcal{I}$ multidegree homogenous, we get multi-graded Hilbert series

$$\begin{aligned} H(t_1,\dots,t_n) &= \sum_{m\in\mathbb{Z}^n_{\ge 0}} \dim_{\mathbb{C}}(R/\mathcal{I})^{(m_1,\dots,m_n)} t_1^{m_1} t_2^{m_2} \dots t_n^{m_n} \\ &= \frac{k(t_1,\dots,t_n)}{\prod_{i=1}^n \prod_{j=1}^n (1-t_i)} \end{aligned}$$

Next, let's do the change of variable $t_i \mapsto 1 - t_i$, we get the lowest degree part to be the multi-degree of $R/\mathcal{I}$.

Remark 9.18. What we are secretly doing is, we have the left action of diagonal invertible matrices T to (z_{ij}). Then $V(\mathcal{I})$ has an T-equivariant cohomology class which is more or less the multi-degree of $R/\mathcal{I}$ we are considering, and the k-polynomial is more or less the T-equivariant K-class.

Theorem 9.19 (Feher-Rimanyi, 2003; Knutson-Miller, 2005)**.** *The matrix Schubert variety X_w has multi-degree is equal $\mathfrak{S}_w(t_1,\dots,t_n)$. Next, sending $t_i \mapsto 1 - t_i$, the k-polynomial becomes $\overline{\mathfrak{S}_w}(\beta = -1, t_1, \dots, t_n)$.*

Well, it is natural to ask, we got 8 different bases of ASym, do we have analogous varieties for the rest seven bases? We don't know.

10 Grobner Bases

The next topic will be Grobner bases.

Definition 10.1. Let $R = \mathbb{C}[Z]$. A ***monomial order*** is a total orders on monomials such that:

1. $m < n$ if and only if $mp < np$.
2. $m \leq mp$.

Example 10.2. The ***lexicographic order*** is a monomial order: we order the variables $z_1 < z_2 < \dots < z_n$, then $Z^a < Z^b$ if rightmost non-zero entry of $b - a$ is positive.

The ***graded lexicographic order*** is also monomial order: first order by total degree, break ties with lexicographic order.

Definition 10.3. For $f \in R$, the ***initial term*** $\operatorname{init}(f)$ is the one term in f whose monomial is biggest. For $F \subseteq R$, we define $\operatorname{init}(F) = \{\operatorname{init}(f) : f \in F\}$, and hence for an ideal $\mathcal{I}$ we define the ***initial ideal*** as $\langle \operatorname{init}(\mathcal{I}) \rangle$.

Example 10.4. Consider two variables $x < y$ and $f = 2x^2 + 3xy + 4y^2$, then we see $V(f)$ is a hyperbola. For $t \in \mathbb{C}\backslash 0$, we send $(x, y) \mapsto (tx, t^2 y)$, then we get $f \mapsto 2t^2x^2 + 3t^3xy + 4t^4y^2$. Then, observe

$$\begin{aligned} V(t \cdot f) &= V(2t^2x^2 + 3t^3xy + 4t^4y^2) \\ &= V(\frac{2t^2x^2 + 3t^3xy + 4t^4y^2}{t^4}) \\ &= V(\frac{2}{4t^2}x^2 + \frac{3}{4t}xy + y^2) \end{aligned}$$

However, if we consider $t \to \infty$, we get our vanishing locus becomes $V(y^2)$. To think about this, we have the two branches of the hyperbola becomes closer and closer together as t becomes bigger and bigger, and when t goes to infinity, the two branches just become one branch with multiplicity 2.

The moral is, if you don't understand what hyperbolas are, we can just look at the limit of this family of hyperbolas, and we see it is $V(y^2)$ in our case, and we can say, oh, so hyperbolas are "just" two lines. So, using degeneration, we get simpler objects, and properties can only get worse.

Remark 10.5. The general principal is, for many properties, $V(\langle \operatorname{init} \mathcal{I} \rangle)$ is weakly "worse" than $V(\mathcal{I})$. So if you can compute $V(\langle \operatorname{init} \mathcal{I} \rangle)$ is "nice", then $V(\mathcal{I})$ is at least that nice.

Here are some examples:

1. If $\langle \operatorname{init} \mathcal{I} \rangle$ is reduced, then $\mathcal{I}$ was reduced.
2. If $\langle \operatorname{init} \mathcal{I} \rangle$ is prime, then $\mathcal{I}$ was prime.
3. If $\langle \operatorname{init} \mathcal{I} \rangle$ is Cohen-Macaulay (CM), then $\mathcal{I}$ was CM.
4. The multi-graded Hilbert function of $R/\mathcal{I}$ and $R/\langle \operatorname{init} \mathcal{I} \rangle$ are equal. So, the k-polynomial and the multidegree also match.

Thus, this gives us a way to compute the multidegree of matrix Schubert variety, by finding $\langle \operatorname{init} \mathcal{I}_w \rangle$.

Now, suppose $\mathcal{I} = \langle g_1, ..., g_k \rangle$, then we can take $\langle \operatorname{init}(g_i) : 1 \le i \le k \rangle \subseteq \langle \operatorname{init} \mathcal{I} \rangle$. It would be nice if they are equal.

Definition 10.6. Say $\{g_1, ..., g_k\}$ is a ***Grobner basis*** for $\mathcal{I}$ if $\mathcal{I} = \langle g_1, ..., g_k \rangle$ and

$$\langle \operatorname{init} g_i : 1 \le i \le n \rangle = \langle \operatorname{init} \mathcal{I} \rangle$$

Example 10.7. Here is an example of non-Grobner basis. Say $w = 2143$, and our Fulton generators is

$$\mathcal{I}_w = \langle z_{11}, d := \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \rangle$$

Thus we see $z_{11}z_{22}z_{33} \in \mathcal{I}$ and $z_{11}z_{23}z_{32} \in \mathcal{I}$. Then $d - z_{11}z_{22}z_{33} + z_{11}z_{23}z_{32} \in \mathcal{I}$ and this term contains no z_{11}. Then, suppose take a monomial order such that $z_{11}z_{22}z_{33}$ is the lead term for d.

THen, we see $\operatorname{init}(\mathcal{I}_w)$ is not the same as $\operatorname{init}(z_{11}) = z_{11}$ and $\operatorname{init}(d) = z_{11}z_{22}z_{33}$ because $\operatorname{init}(d) \in \langle z_{11} \rangle$ and hence

$$\langle \operatorname{init}(z_{11}), \operatorname{init}(d) \rangle = \langle z_{11} \rangle \subsetneq \langle \operatorname{init} \mathcal{I}_w \rangle$$

This shows, by picking the monomial order in a bad way, we get a non-Grobner basis.

Last time we talked about Grobner bases. However, we note this is not realy a basis because we can always add new terms to a Grobner basis to get another, i.e. we don't have minimal condition on the basis, hence they are realy just a spanning set.

Next, we ask do they exist? Well, of course, just take the whole $\mathcal{I}$. But this is bad, so a better question is, do finite Grobner basis exists?

Theorem 10.8 (Hilbert). *Every $\mathcal{I}$ has a finite set of generators.*

Theorem 10.9 (Buchberger). *There is a finite algorithm to produce a finite Grobner basis from finite set of generator.*

However, Hilbert's basis theorem only tells you finite generators exists, but not what they are. So, in general it is still hard to find Grobner basis.

Definition 10.10. For $R = \mathbb{C}[Z]$ where Z is $n \times n$ matrix of variables. We say a monomial order is ***diagonal*** if the lead term of every minor is the product along the main diagonal. We say it is ***anti-diagonal*** if lead term of every minor is product along the anti-diagonal.

Theorem 10.11 (Knutson-Miller, 2005). *For all $w \in S_n$, and any anti-diagonal monomial order, the Fulton generators are a Grobner basis.*

Corollary 10.11.1. *$\mathcal{I}_w$ is radical.*

Proof. Fulton generators are determinants, so lead terms are square-free and hence there are no root we can take. This shows $\mathcal{I}_w$ is radical as well. ♡

Example 10.12. Last time we considered $\operatorname{init}(\mathcal{I}_{2143})$, which gives non-Grobner basis. Now we consider the anti-diagonal one, which gives

$$\langle \operatorname{init}_{ad}(\mathcal{I}_{2143})\rangle = \langle z_{11}, z_{13}z_{22}z_{31}\rangle = \langle z_{11}, z_{13}\rangle \cap \langle z_{11}, z_{22}\rangle \cap \langle z_{11}, z_{31}\rangle$$

Viz, it has pretty simple primiary decomposition (i.e. you get prime ideals).

Theorem 10.13 (Lasker, 1905; Noether, 1921). *Every radical ideal $\mathcal{I}$ in a polynomial ring $\mathbb{C}[Z]$ has an essentially unique primiary decomposition into prime ideals, i.e. $\mathcal{I} = \mathcal{P}_1 \cap \ldots \cap \mathcal{P}_k$.*

Geometrically, this tells us $V(\mathcal{I}) = V(\mathcal{P}_1 \cap \ldots \cap \mathcal{P}_k) = V(\mathcal{P}_1) \cup \ldots \cup V(\mathcal{P}_k)$. Hence, in the example of $\mathcal{I}_{2143}$, see

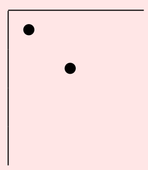
$$V(\mathcal{I}_{2143}) = V(z_{11}, z_{13}) \cup V(z_{11}, z_{22}) \cup V(z_{11}, z_{31})$$

But we see $V(z_{11}, z_{13})$ is just the codimension 2 linear subspace given by $z_{11} - z_{31} = 0$, and similarly for the other two.

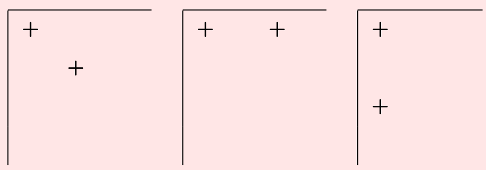
We can always do this, and decompose matrix Schubert varieties into union of coordinate subspaces.

11 Pipe Dreams, Again

Recall $w = 2143$, and let's consider its pipe dream. To get the pipe dream, we first take its Rothe diagram



Thus, we get all the pipe dreams by slide the + along the diagonal its located. Hence we get three pipe dreams as follows



But, now if we look at where the crosses are located, they are exactly given by $((1,1),(2,2))$, $((1,1),(3,1))$ and $((1,1),(1,3))$. This is exactly our coordinate subspaces, and this is always happening for matrix Schubert varieties.

Theorem 11.1 (Knutson, 2019). *Schubert polynomials, pipe dreams, and multidegree of matrix Schubert varieties all satisfies the same "cotransition" recurrence as follows: for $w \neq w_0$, let i be the least such that $i + w(i) < n$, then*

$$x_i \mathfrak{S}_w = \sum_{\substack{w \lessdot u \\ w(i) \neq u(i)}} \mathfrak{S}_u$$

Clearly they have the same base case:

1. $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \ldots x_n^0$
2. We only have one pipe dream for w_0, which is all + stuffed at the top left corner.
3. The matrix Schubert variety $\mathcal{I}_{w_0} = \langle z_{11}, z_{12}, z_{13}, z_{21}, z_{22}, z_{31}, \ldots \rangle$

Next, we check this on $w = 2143 = s_1 s_3 = s_3 s_1$. In this case, the Rothe diagram is

$$\begin{array}{|cccc}
\hline
\bullet & \color{red}{\star} & & \\
\star & & & \\
 & & \bullet & \star \\
 & & \star &
\end{array}$$

where the red star is the i we are looking for, i.e. $i = 1$ and $w(i) = 2$.

Next, we check cover in the Bruhat order. They are given by

$$u_1 = s_1 s_3 s_2 = 2143 s_2 = 2413$$

$$u_2 = s_2 s_1 s_3 = s_2 2143 = 3142$$

$$u_3 = s_1 s_2 s_3 = 2341$$

$$u_4 = s_3 s_2 s_1 = 4123$$

In the above, u_1, u_3 does not work because we don't have $w(1) \neq u(1)$. Thus, by the cotransition recurrence, we have

$$\mathfrak{S}_{2143} = x_1(x_1 + x_2 + x_3)$$

On the other hand,

$$\mathfrak{S}_{3142} = x_1^2(x_2 + x_3)$$

$$\mathfrak{S}_{4123} = x_1^3$$

It's not hard to see we indeed have

$$x_1 \mathfrak{S}_{2143} = \mathfrak{S}_{3142} + \mathfrak{S}_{4123}$$

Theorem 11.2 (Knutson-Miller, 2005). *For $w \in S_n$,*

$$\langle \mathrm{init}_{ad}(\mathcal{I}_w) \rangle = \bigcap_{\substack{\text{reduced} \\ \text{pipe dreams} \\ P \text{ of } w}} \langle z_{ij} : P \textit{ has } \boxplus \textit{ in } (i,j) \rangle$$

We note on the left hand side, we need to pick a particular order (i.e. anti-diagonal order). This means, when we change the monomial order on the LHS, we should have different combinatorics on the right hand side.

In particular, what about diagonal orders? In this case, the Fulton generators for $w = 2143$ were not Grobner basis, so we have to try harder.

Theorem 11.3 (Knutson-Miller-Yong, 2009). *The Fulton generator of $\mathcal{I}_w$ are diagonal Grobner iff w avoid 2143 (i.e. w is* ***vexillary****).*

The terms of the prime decomposition of $\langle \mathrm{init}_d \mathcal{I}_w \rangle$ correspond to flagged semistandard tableaux. In particular, if w is Grassmannian permutation, then this correspond to tableaux formula for Schur functions.

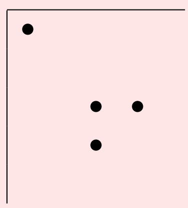
This seems like the end of the story, but we also have an ester egg from Lascoux.

Definition 11.4 (Hamaker-Pechenik-Weigandt). The ***dominant part***, $\mathrm{Dom}(w)$, of the Rothe diagram $D(w)$ is its cells with rank 0. Let Z be a matrix of variables, let $Z^{\mathrm{Dom}(w)}$ be Z with $Z_{ij} \mapsto 0$ if $(i,j) \in \mathrm{Dom}(w)$.

Example 11.5. In the case of 2143, we get

$$Z^{\mathrm{Dom}(w)} = \begin{bmatrix} 0 & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}$$

On the other hand, if $w = 21543$, then the Rothe diagram is

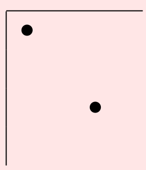


Then $Z^{\mathrm{Dom}(w)}$ is similar to the above, with only z_{11} equal 0.

Then, we have $\mathcal{I}$ is equal the ideal generated by z_{ij} for $(i,j) \in \mathrm{Dom}(w)$ and the $(r_{ij}+1) \times (r_{ij}+1)$ minors in NW submatrix of $Z_{ij}^{\mathrm{Dom}(w)}$ for essential, non-dominant (i,j). Those are called the ***CDG (Canca, DeNegr, Corla) generators***.

Last time we have the dominant part associated with the Rothe diagram.

Example 11.6. Consider $w = 2143$ with Rothe diagram



Then we get

$$\mathcal{I}_{2143} = \left\langle z_{11}, \begin{vmatrix} 0 & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

and

$$\mathcal{I}_{2143} = \langle z_{11}, z_{12}z_{23}z_{31} + z_{13}z_{21}z_{32} - z_{12}z_{21}z_{33} - z_{13}z_{22}z_{31} \rangle$$

The anti-diagonal Grobner basis is $z_{11}, z_{13}z_{22}z_{31}$, and the diagonal Grobner basis is $z_{11}, z_{12}z_{21}z_{33}$.

Theorem 11.7 (Klein, 2020)**.** *The CDG generators of $\mathcal{I}_w$ are diagonal Grobner iff we avoids*

$$13254, 21543, 214635, 215364, 241635, 315264, 215634, 4261735$$

In the above, we note 214635 and 215364 are inverse of each other, and 241645 and 315264 are inverse of each other.

In the case of 13254, it is just 2143 bumped to S_5 by adding a one.

In the case of 21543, even we have a box at the top left place, it is still bad because the two boxes $(4,3)$ and $(3,4)$ locations.

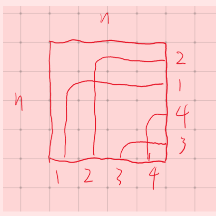
In the case of 214635, there are two boxes on the same row.

Next, we are going to talk about bumpless pipe dreams.

Consider the tiling of n by n grids, and last time we have two tiles, and for bumpless pipe dreams we are going to throw away the bumps and use the following set of tiles ⊞, ◺, ◸, □, ◫, ⊟.

The rules are, there are n pipes starting at the bottom edges and ending at the right edge, each pair of pipes crossing at most once.

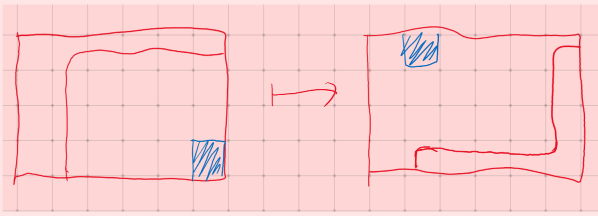
Here is an example:



Each BPD has an associated permutation, as we can see from above.

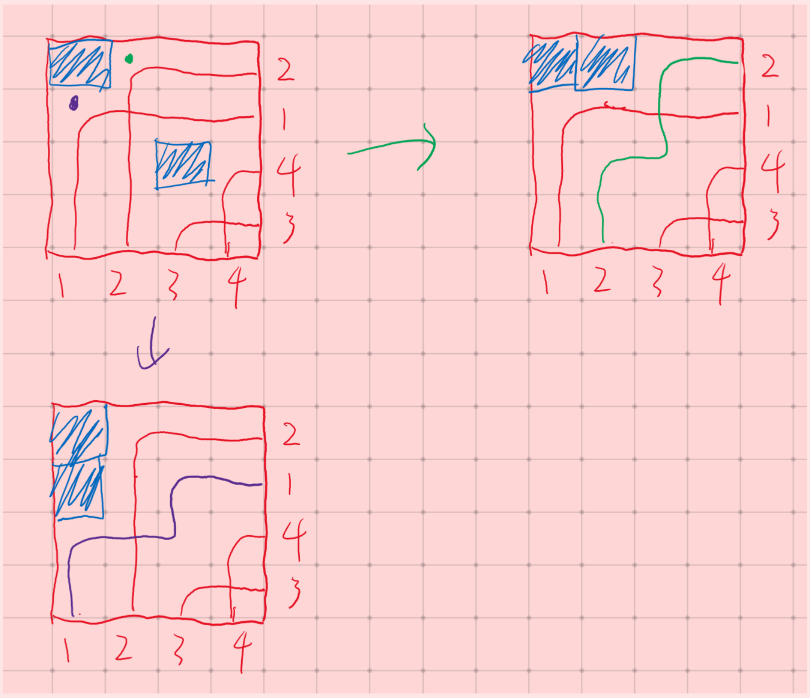
The BPDs for $w \in S_n$ are constructed by droop moves. First, each $w \in S_n$ associates to a natural BPD by change the laser dots in the Rothe diagram to the ◺-tile. We can see from the above example. Next, we are going to get all the BPDs from this via droop moves.

A droop move is given by the following:



where the blue colour indicates blank tiles.

Let's see a real example in action:



where on the blue indicates blank, the green and purple dots indicates possible droops, and the left and bottom are results of droops.

Now let's look back to the CDG generator of $\mathcal{I}_{2143}$, which is

$$\langle z_{11}, z_{12}z_{21}z_{33}\rangle = \langle z_{11}, z_{12}\rangle \cap \langle z_{11}, z_{21}\rangle \cap \langle z_{11}, z_{33}\rangle$$

How is this correspond to our above BPD? Well, they correspond to the blank tiles.

Theorem 11.8 (Lam-Lee-Shimozono, 2021; Lascoux, 2002; Weigandt, 2021)**.** *For all $w \in S_n$,*

$$\mathfrak{S}_w(x, y) = \sum_{P \in \mathrm{BPD}(w)} \prod_{\substack{(i,j)\in P \\ \text{is } \square}} (x_i - y_j)$$

Let's compare the oridinary formula and this new formula. The old one we have

$$\mathfrak{S}_w(x,y) = \sum_{P \in \mathrm{PD}(w)} \prod_{\substack{(i,j) \in P \\ \text{is } \boxplus}} (x_i - y_j)$$

and we see they are very similar.

However, when we write out the example, we see the new formula gives

$$\begin{aligned}
\mathfrak{S}_{2143}(x,y) &= (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2) \\
&= (x_3 - y_3) + (x_2 - y_1) + (x_1 - y_2)
\end{aligned}$$

and the old formula gives

$$\begin{aligned}
\mathfrak{S}_{2143}(x,y) &= (x_1 - y_1)(x_3 - y_1) + (x_1 - y_1)(x_2 - y_2) + (x_1 - y_1)(x_1 - y_3) \\
&= (x_3 - y_1) + (x_2 - y_2) + (x_1 - y_3)
\end{aligned}$$

We see they are the same polynomial, with variables changed. This implies there is no bijection between BPD(w) and PD(w) preserving the double weight.

> **Theorem 11.9** (Gao-Huang, 2021). *There is an explicit bijection preserving the x-weight.*

The naive hope is that, for $w \in S_n$, $\langle \mathrm{init}_d(\mathcal{I}_w) \rangle$ has a prime decomposition as

$$\bigcap_{P \in \mathrm{BPD}(w)} \langle z_{ij} : (i,j) \in P \text{ is } \square \rangle$$

This works for $w = 2143$, but this is wrong in general!

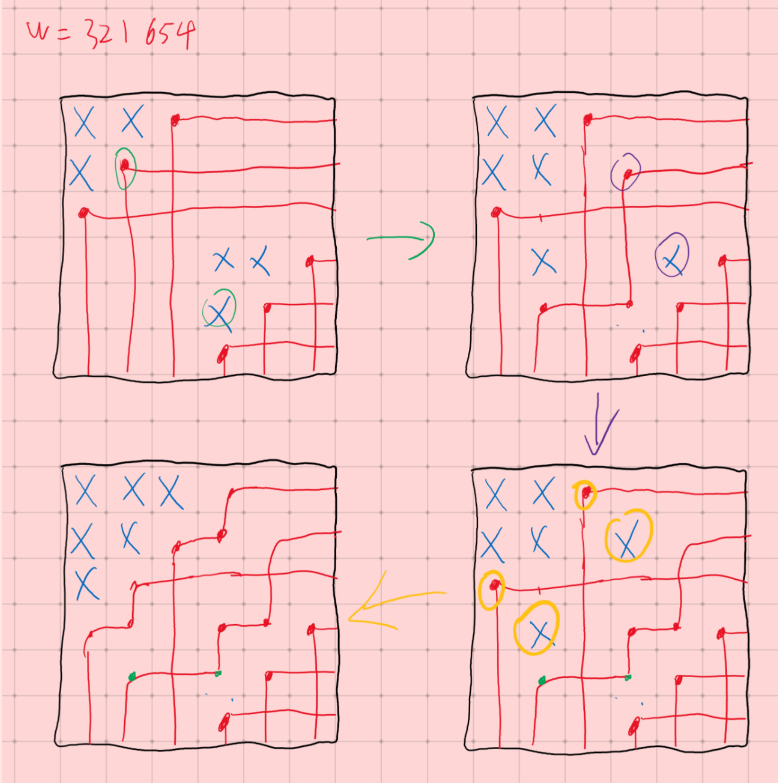
Problem one: In $\langle \mathrm{init}_d(\mathcal{I}_w) \rangle$ is not radical in general. Thus it does not have a prime decomposition at all (it still has primiary decomposition, of course).

Problem two: In general, $\langle \mathrm{init}_d(\mathcal{I}_w) \rangle$ is not well-defined. It depends on the choice of diagonal order.

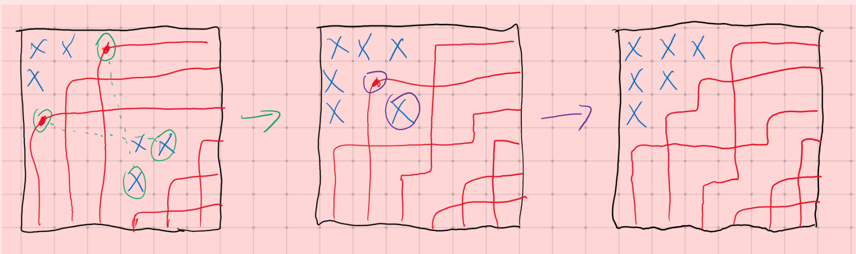
The resolution of problem two is that, the dependence is mild, although we get different primiary decomposition, the set of associated primes are equal. Actually, the multiplicity of each associated prime is equal as well (in this case, we can't distinguish, say in $\mathbb{C}[x,y,z]$, that $\langle x^2, y \rangle$ and $\langle x, y^2 \rangle$ are different)

The resolution of problem one is that, BPDs also appear with multiplicity. What this means is that, in normal pipe dream, if we know all the crosses, then we know the entire tiling, but this is not the same in BPDs, where even we know all the blank tiles, we can't ascertain a unique BPD.

> **Example 11.10.** For example, say $w = 321654$. Then one possible list of droop moves are



Another possible sequence that gives the same resulting BPD is



Both contribute to

$$(x_1 - y_1)(x_1 - y_2)(x_1 - y_3)(x_2 - y_1)(x_2 - y_2)(x_3 - y_1)$$

This correspond to coordinate subspace

$$\langle z_{11}, z_{12}, z_{13}, z_{21}, z_{22}, z_{31} \rangle$$

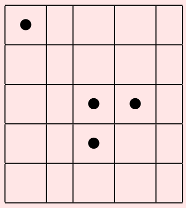
But every diagonal degeneration of $\mathcal{I}_{321654}$ has the above subspace with multiplicity 2.

Theorem 11.11 (Klein-Weigandt, 2021)**.** *For any diagonal monomial order d and any $w \in S_n$, the irreducible components of $V(\langle \mathrm{init}_d(\mathcal{I}_w) \rangle)$ counted with multiplicity correspond to the BPDs for w counted with multiplicity, i.e. the coordinate subspace $V(\langle z_{i_1,j_1}, \ldots, z_{i_k,j_k} \rangle)$ appear with multiplicity m if and only if* *there exist m many BPDs for w with $\square$ in exactly positions (i_l, j_l).*

There are still some problems with this:

1. We still don't have Grobner bases (except CDG case).
2. When does multiplicity occur?

Example 11.12. Consider $w = 21543$, which is not CDG. The Rothe diagram is

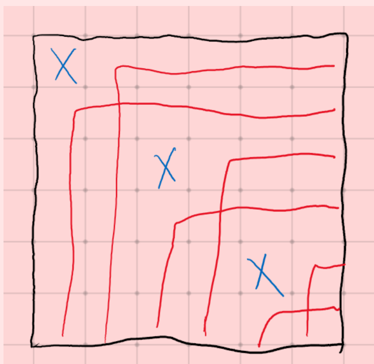


The Fulton generators are z_{11}, four 3×3 minors in the matrix enclosed by z_{11} and z_{34}, and three more 3×3 minors enclosed by z_{11} and z_{43}. Take diagonal order that is going left to right in rows first and top to bottom second. Then

$$\langle \mathrm{init}_d(\mathcal{I}_{21543}) \rangle$$

has a minimal generating set with q elements including one with degree 5. But what is the degree five element? We don't realy know...

Example 11.13. Consider $w = 214365$. Let d be diagonal order that's lexicographic and it's going left to right first and top to bottom. Let d' be lexicographic and going top to bottom first, then going left to right second. The Rothe diagram for w is



Then the Fulton generators are z_{11}, $\det(z_{11} \to z_{33})$ and $\det(z_{11} \to z_{55})$ where $a \to b$ means the submatrix enclosed by a and b. Then $\langle \mathrm{init}_d(\mathcal{I}_{214365}) \rangle$ is generated by $z_{11}, z_{12}z_{21}z_{33}$, $z_{12}z_{21}z_{34}z_{43}z_{55}$, $z_{12}z_{23}z_{31}z_{34}z_{43}z_{55}$ and $z_{13}z_{21}^2z_{32}z_{31}z_{43}z_{55}$. When we take d' instead of d, we get $\langle \mathrm{init}_{d'}(\mathcal{I}_{214365}) \rangle$ is generated almost similar terms, namely $z_{11}, z_{12}z_{21}z_{33}$, $z_{12}z_{21}z_{34}z_{43}z_{55}$, $z_{12}^2z_{23}z_{31}z_{34}z_{43}z_{55}$ and $z_{13}z_{21}z_{32}z_{31}z_{43}z_{55}$. Here, the last generator don't have square terms anymore, and the second last generator gained a square term.

Both gives $V(\langle z_{11}, z_{12}, z_{21}\rangle)$ with multiplicity 2, one from $V(z_{11}, z_{12}^2, z_{21})$ and one from $V(z_{11}, z_{12}, z_{21}^2)$.

Example 11.14. Now move to $w = 2143675$. Define d and d' as as before. Then $\langle \mathrm{init}_d(\mathcal{I}_w)\rangle$ has 43 associated primes at height 4 that correspond to irreducible components and 10 embedded primes at height 5. On the other hand, if we change d to d', then we still get 43 associated primes at height 4, and 6 embedded primes at height 5.

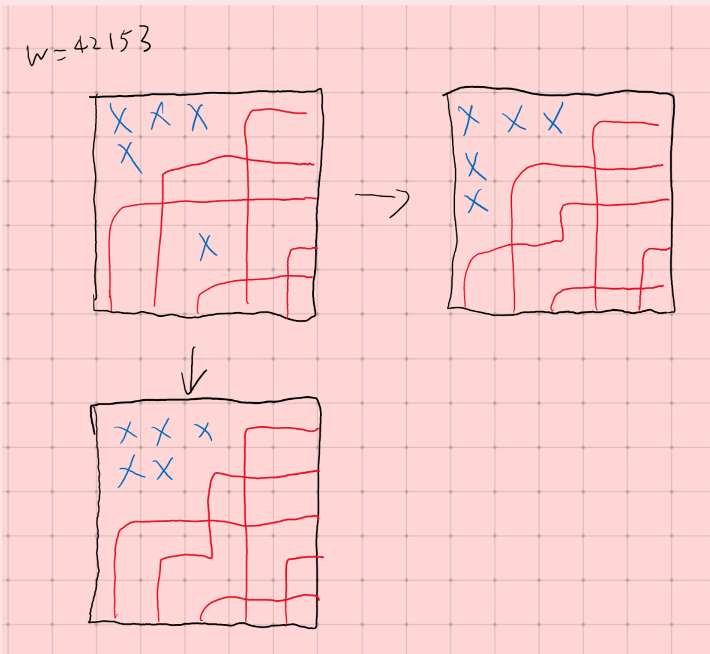
We don't have any understanding when is this happening (i.e. when do embedded primes occur?). It would be nice if this is some pattern avoiding, but we don't know.

Next, we talk about how transition fit in the story.

Theorem 11.15 (Equivariant Transition, Kohnert-Veigneau, 1997)**.** *Let (a, b) be a maximally southeast box of $D(w)$. Set $v = wt_{a,w^{-1}(b)}$ where $t_{a,b}$ is the permutation swaps a and b such that $D(w) = D(v) \cup \{(a, b)\}$. Let $\Phi(w, v) = \{vt_{i,a} : i < a, vt_{i,a} > v\}$. Then*

$$\mathfrak{S}_w(x, y) = (x_a - y_b)\mathfrak{S}_v(x, y) + \sum_{u \in \Phi(w,v)} \mathfrak{S}_u(x, y)$$

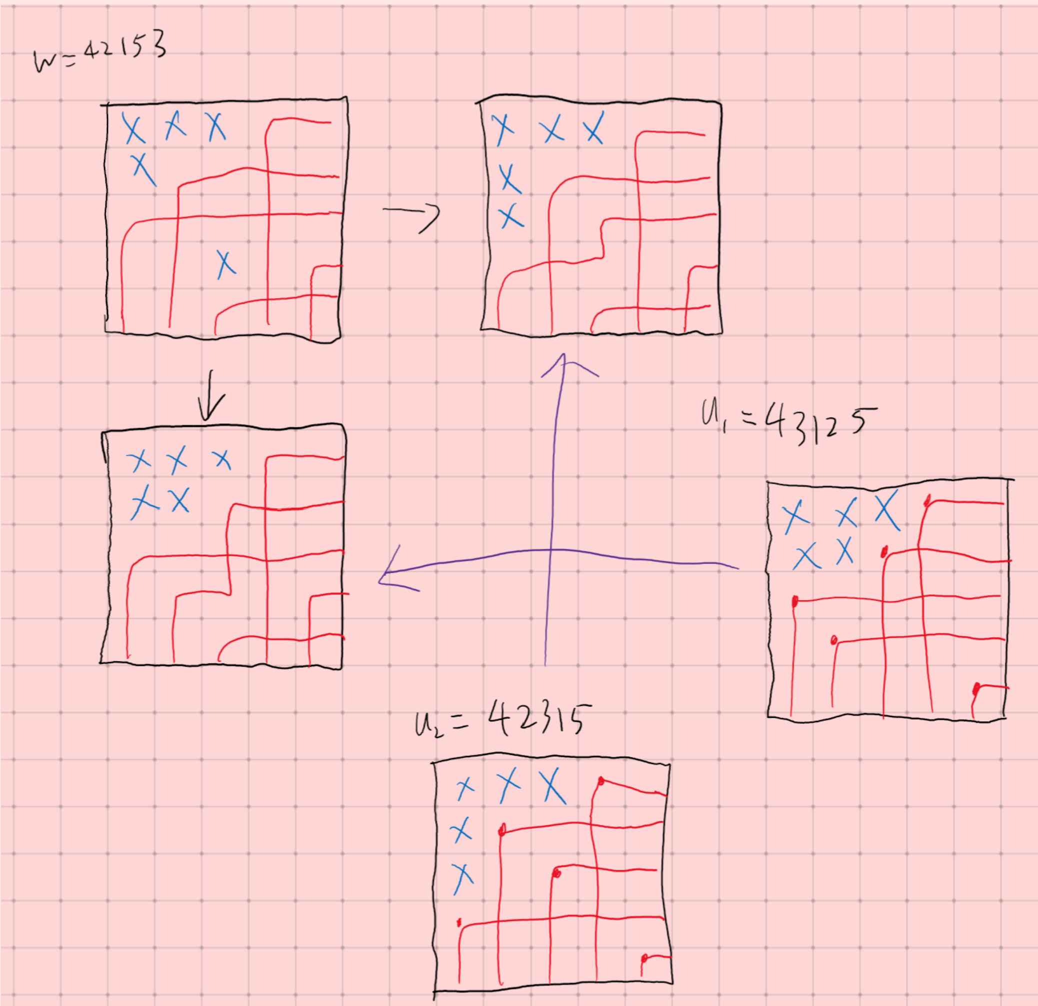
Example 11.16. Let $w = 42154$, then we have the following droops:



Next, fix $(a, b) = (4, 3)$, and let $v = wt_{45} = 42135$. Then we get

$$\Phi(w, v) = \{u_1 := 43125 = vt_{24}, u_2 := 42315 = vt_{34}\}$$

For u_1 and u_2, we have the following



where the purple arrow indicates equality. Thus, we see the above theorem indeed holds if we add $v = wt_{45}$ in the picture.

Theorem 11.17 (Weigandt, 2021). *Let (a, b) be maximally southeast in $D(w)$ and $v = wt_{a,w^{-1}(b)}$. There is a bijection*

$$\Psi : \mathrm{BPD}(w) \to \mathrm{BPD}(v) \uplus \bigcup_{u \in \Phi(w,v)} \mathrm{BPD}(u)$$

such that:

1. *if $\Psi(P) \in \mathrm{BPD}(v)$, then the blank boxes $\square$ for P is equal the blank boxes of $\Psi(P)$ union a blank box at (a, b).*
2. *otherwise the blank boxes of P is equal blank boxes of $\Psi(P)$.*

Proof. The Rothe diagram for v comes from the Rothe diagram for w by supplying the exits of pipes b and $w(a)$. Every BPD for w has the same tiles southeast of (a, b) becausethey are connected by droops, so we can do this same swap on all of BDP(w). Let $P \in \mathrm{BPD}(w)$, then (a, b) has either $\square$ or ⌜.

Case 1: P has $\square$ in (a,b). Then the pipe swap gives a BPD P' for v with $\square(P) = \square(P') \cup \{(a,b)\}$, set $\Psi(P) = P'$. Here $\square(P)$ just mean the blank boxes of P.

Case 2: P has $\boxslash$ in (a,b). Then the pipe swap puts a bumping tile $\boxslash$ in (a,b). But this is a bump and hence we want to fix this, and we have a unique way to do this: replace $\boxslash$ with $\boxplus$ and call the result $\Psi(P)$. Clearly $\square(P) = \square(\Psi(P))$. Now $\Psi(P) \in \bigcup_{u\in\Phi(w,v)} \mathrm{BPD}(u)$.

It remains to check this is a bijection, and the proof follows. ♡