Contents

Chapter 1

Sites And Topos

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1.1 Introduction

So, before we go real, let's answer the question, "why stacks"?

To answer this question, we back up and look ad moduli spaces first. Frequently, when studying geometric objects (such as enumerating genus g curves, Abelian varieties, vector bundles, etc), it helps to look at the "space of all such objects".

When that space exists, it is called a moduli space.

Why is this helpful?

Example 1.1.1

How many plane conics (i.e. a degree 2 curve in $\mathbb{P}^2$) go through 5 general points?

We can prove there exists a unique conic. To do this, we consider the moduli space of conics in $\mathbb{P}^2$. A degree 2 curve $C \subseteq \mathbb{P}^2$ (with coordinate x, y, z) correspond bijectively to $C = V(ax^2 + by^2 + cz^2 + dxy + exz + fyz)$. Observe if we take (a, b, c, d, e, f) and scale it by $\lambda \neq 0$ in field k, then we see we still get the same conic C. In particular, this is the only relation we need to mod out, i.e. two conics C, C' are the same iff their coefficients differ by a scalar.

Thus moduli space of plane conics are given by $\{(a,b,c,d,e,f)\}/k^*$, which is exactly $\mathbb{P}^5_k$ with coordinates (a,b,c,d,e,f).

Now, for a fixed point $p=(x_0:y_0:z_0)$ in $\mathbb{P}^2$, a conic $C:=V(f)$ with $f=ax^2+by^2+cz^2+dxy+exz+fyz$ contains p iff $f(x_0,y_0,z_0)=0$. To translate this condition to a condition on a,b,c,d,e,f, we see this becomes linear constraint on a,b,c,d,e,f, i.e. $H_p:=\{C\in\mathbb{P}^5 : p\in C\}$ is a hyperplane.

Thus, choosing 5 general points $p_1,\ldots,p_5$, we see $\{C\in\mathbb{P}^5 : p_1,\ldots,p_5\in C\}=H_{p_1}\cap\ldots\cap H_{p_5}$. But since the points p_i are general, we have $H_{p_1}\cap\ldots\cap H_{p_5}$ is exactly one point, i.e. there is a unique conic that does the job.

The above toy example shows it is helpful to consider moduli spaces. This is all good and sound, but we have a problem.

Moduli spaces rarely exists, or, they are rarely schemes. They are typically stacks!

Thus, in order to use moduli spaces, we are forced to consider stacks. Well, there are ways to get around this, but we lose information along the way (e.g. GIT).

So, why aren't moduli spaces schemes?

Example 1.1.2

Let $\mathcal{M}$ be the moduli space of vector bundles, i.e. a map $X\to\mathcal{M}$ is equivalent to giving vector bundle $\mathscr{E}$ on X. Suppose it is a scheme. Then, lets consider two maps $\mathbb{P}^1\to\mathcal{M}$, the first one given by $\mathbb{P}^1\xrightarrow{\mathscr{O}}\mathcal{M}$ and the second one given by $\mathbb{P}^1\xrightarrow{\mathscr{O}(1)}\mathcal{M}$.

Then, since $\mathbb{P}^1=U_1\cup U_2$ with $U_i=\mathbb{A}^1$, we get

$$\begin{tikzcd} U_1 \arrow[d, "\subseteq"] \arrow[dr, "\mathscr{O}(1)|_{U_1}"] & \\ \mathbb{P}^1 \arrow[r, "\mathscr{O}(1)"] & \mathcal{M} \end{tikzcd}$$

where $\mathscr{O}(1)|_{U_1}\cong\mathscr{O}|_{U_1}$. On the other hand, we also have

$$\begin{tikzcd} U_2 \arrow[d, "\subseteq"] \arrow[dr, "\mathscr{O}(1)|_{U_2}"] & \\ \mathbb{P}^1 \arrow[r, "\mathscr{O}(1)"] & \mathcal{M} \end{tikzcd}$$

where $\mathscr{O}(1)|_{U_2}\cong\mathscr{O}$ as well. However, if $f|_{U_1}=g|_{U_1}$ and $f|_{U_2}=g|_{U_2}$, and if $\mathcal{M}$ were a scheme, then $f=g$, i.e. in our case we get $\mathscr{O}\cong\mathscr{O}(1)$ on $\mathbb{P}^1$, which is a contradiction.

So, the intuition for stacks.

With a moduli space (e.g. plane conic), a map $X\to\mathcal{M}$ corresponds to some geometric object on X. If $\mathcal{M}$ is a scheme, then $\mathrm{Hom}_{(\mathbf{Sch})}(X,\mathcal{M})$ is a set. For stack $\mathcal{M}$, the set $\mathrm{Hom}(X,\mathcal{M})$ will be a category. For example, in the case of vector bundles,

$\mathrm{Hom}(X,\mathcal{M})$ just become the category of vector bundles (over X). In this case, if two maps f,g agree on open cover, they might not be the same map, hence the problem occured in the above example is throw out of the window.

To define stacks, there are two main points:

1. Pure category theory (which is roughly analogous to a sheaf, and we call this "categorical stacks"). The main input here is what's called a fibered category.
2. Geometry (which is roughly extra constraints on the "categorical stacks"). This additional geometry makes them algebraic stacks.

1.2 A Little Bit Category Theory

Recall a presheaf on a topological space X is a choice of sets $\mathscr{F}(U)$ for all $U \subseteq X$ open, and a choice of restriction maps $\rho^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$ whenever $V \subseteq U$, with a compatibility condition: if $W \subseteq V \subseteq U$ then $\rho^V_W \rho^U_V = \rho^U_W$.

We want to reformulate this definition. Another way to saying this is: consider the category $\mathrm{Op}(X)$ whose objects are open subsets $U \subseteq X$, and arrows are inclusions, i.e. $V \to U$ iff $V \subseteq U$. Then a presheaf $\mathscr{F}$ is a functor $\mathrm{Op}(X)^{\mathrm{opp}} \to (\mathbf{Sets})$.

Definition 1.2.1

If $\mathcal{C}$ is any category, a ***presheaf on*** $\mathcal{C}$ is defined by

$$\mathrm{Pre}(\mathcal{C}) := \hat{\mathcal{C}} := \mathrm{Fun}(\mathcal{C}^{\mathrm{opp}}, (\mathbf{Sets}))$$

where $\mathrm{Fun}(\mathcal{C},\mathcal{B})$ is the collection of functors.

Theorem 1.2.2: Yoneda's Lemma

Let $\mathcal{C}$ be a category. Then

$$\mathcal{C} \to \mathrm{Pre}(\mathcal{C})$$
$$X \mapsto h_X := \mathrm{Hom}(-,X)$$

is an embedding, i.e. if $h_X \cong h_Y$ then $X \cong Y$, and

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) = \mathrm{Hom}_{\hat{\mathcal{C}}}(h_X,h_Y) = \{\textit{natural trans } h_X \to h_Y\}$$

In fact, we have $\mathrm{Hom}(h_X,\mathscr{F}) = \mathscr{F}(X)$ if $\mathscr{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{opp}}, (\mathbf{Sets}))$.

Before we give a sketch proof, we note this is useful because it allows us to put $\mathcal{C}$ and $\hat{\mathcal{C}}$ objects into a same diagram. For example, rather than writing $h_X \to \mathscr{F}$, we can write $X \mapsto \mathscr{F}$. This is handy, say, given $f \in \mathscr{F}(X) \in (\mathbf{Sets})$ with $\mathscr{F} \in \hat{\mathcal{C}}$, we know that this corresponds to some arrow $h_X \to \mathscr{F}$, and hence we can write $X(f)$, instead of $h_X(f)$. This is good because, say we have $Y \xrightarrow{F} X$, then let $f \in \mathscr{F}(X)$, we get $F^*(f) \in \mathscr{F}(Y)$. Then we can write a diagram

$$\begin{tikzcd} Y \arrow[r, "F"] \arrow[dr, "F^*(f)"'] & X \arrow[d, "f"] \\ & \mathscr{F} \end{tikzcd}$$

In practice, we might want $\mathscr{F}$ be to a moduli space. Then Yoneda says that $X \to \mathscr{F}$ is the same as $\mathscr{F}(X)$.

Now lets give a sketch proof.

Proof. Notice if $\mathscr{F}(X) = \mathrm{Hom}(h_X, \mathscr{F})$, then in particular, $\mathrm{Hom}(h_X, h_Y) = h_Y(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Thus we just need to show $\mathscr{F}(X) = \mathrm{Hom}(h_X, \mathscr{F})$. It suffices to show how to go back and forth. Suppose we are given $h_X \xrightarrow{\eta} \mathscr{F}$, we get

$$\eta(X) : h_X(X) = \mathrm{Hom}(X, X) \to \mathscr{F}(X)$$

In particular we get $\mathrm{Id} \in \mathrm{Hom}(X, X)$ and it correspond to

$$(\eta(X))(\mathrm{Id}) \in \mathscr{F}(X)$$

Viz, given $h_x \to \mathscr{F}$, we get an element of $\mathscr{F}(X)$. Conversely, given an element $g \in \mathscr{F}(X)$, we get a map $\eta : h_X \to \mathscr{F}$ defined by, for all $Y \in \mathcal{C}$, $\eta(Y) : h_X(Y) \to \mathscr{F}(Y)$, that $f \in h_X(Y)$ is mapped to $\mathscr{F}(f)(g)$, where we note $f : Y \to X$ implies $\mathscr{F}(f) : \mathscr{F}(X) \to \mathscr{F}(Y)$. Next, one need to check $\eta : h_X \to \mathscr{F}$ is a natural transformation, and it is inverse of the first map.

Let's see why this is a natural transformation. To show that, we pick $A \xrightarrow{f} B$ be any arrow in $\mathcal{C}$. Then we get

$$\begin{tikzcd} i : B \to X \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & i \circ f \arrow[ddd, mapsto] \\ & \mathrm{Hom}(B, X) \arrow[r] \arrow[d] & \mathrm{Hom}(A, X) \arrow[d] & \\ & \mathscr{F}(B) \arrow[r] & \mathscr{F}(A) & \\ \mathscr{F}(i)(g) \arrow[rrr, mapsto] & & & \mathscr{F}(i \circ f)(g) \end{tikzcd}$$

where the square commutes because $\mathscr{F}(f) \circ \mathscr{F}(i) = \mathscr{F}(i \circ f)$ as $\mathscr{F} : \mathcal{C}^{\mathrm{opp}} \to (\mathbf{Sets})$ is a functor. We left as an exercise for readers to check they are inverse of each other.

Definition 1.2.3

We say a presheaf $\mathscr{F} \in \hat{\mathcal{C}}$ is ***representable*** (i.e. a "moduli space") if $\mathscr{F} \cong h_X$ for some $X \in \mathcal{C}$.

Using Yoneda's lemma, we can give a characterization of representable presheaves using universal objects.

Definition 1.2.4

Let $\mathscr{F} \in \hat{\mathcal{C}}$. A ***universal object*** for $\mathscr{F}$ is a pair (X, ξ) consisting of an object $X \in \mathcal{C}$, and an element $\xi \in \mathscr{F}(X)$, so that for all $U \in \mathcal{C}$ and $\sigma \in \mathscr{F}(U)$, there is a unique arrow $f : U \to X$ such that $Ff(\xi) = \sigma$.

Proposition 1.2.5

A presheaf $\mathscr{F}$ is representable if and only if it has a universal object.

Proof. The pair (X, ξ) is universal object if the morphism $h_X \to \mathscr{F}$ defined by ξ is an isomorphism. Conversely, every natural transformation is defined by some $\xi \in \mathscr{F}(X)$. This concludes the proof.

In particular, it is not hard to see if $\mathscr{F}$ has universal object (X, ξ), then it is represented by X.

Example 1.2.6

Consider the functor $\mathscr{P} : (\mathbf{Set})^{\text{opp}} \to (\mathbf{Set})$ that sends S to the power set of S. If $f : S \to T$ is a function, then $P(f)$ is defined by $P(f)(\tau) := f^{-1}(\tau) \subseteq S$ for all $\tau \subseteq T$. We show this is representable.

Given a subset $\sigma \subseteq S$, there is a unique function $\chi_\sigma : S \to \{0,1\}$ such that $\chi_\sigma^{-1}(\{1\}) = \sigma$, i.e. the characteristic function of τ. This shows the pair $(\{0,1\}, \{1\})$ is a universal object, and thus $\mathscr{P}$ is represented by $\{0,1\}$.

The above example can be extended to $\mathscr{F} : (\mathbf{Top})^{\text{opp}} \to (\mathbf{Set})$ which sends topological space S to the collection of its open sets. This functor $\mathscr{F}$ is represented by the topological space $\{0,1\}$, endowed with discrete topology.

Example 1.2.7

Let $\mathcal{C} := (\mathbf{HausTop})$ be the category of all Hausdorff topological spaces, and let $\mathscr{F} : \mathcal{C}^{\text{opp}} \to (\mathbf{Set})$ be the functor that sends S to the collection of all its open sets. We claim this $\mathscr{F}$ is not representable, unlike the case for $(\mathbf{Top})^{\text{opp}} \to (\mathbf{Set})$.

To show this, let us assume (X, ξ) is a universal object. Let S be any set, endowed with discrete topology. Then by definition there is a unique continuous function $f : S \to X$ with $f^{-1}(\xi) = S$, i.e. a unique function $S \to \xi$. This means ξ can only have one element. Analogously, there is a unique function $S \to X \backslash \xi$, so $X \backslash \xi$ also has a unique element. This shows X must be a Hausdorff space with two elements, so it must have the discrete topology. Thus ξ is closed in X. Hence, if S is any topological space with a closed subset σ that is not open, then there will be no continuous function $f : S \to X$ with $f^{-1}(\xi) = \sigma$.

Example 1.2.8

Take $(\mathbf{Grp})$ be the category of groups, then $\mathscr{S} \in \widehat{(\mathbf{Grp})}$ that associates each group G the set of all subgroups H is not representable. To see this, suppose we have universal object (Γ, Γ_1). Now take subgroup $\{0\} \subseteq \mathbb{Z}$, then there must be a unique homomorphism $f : \mathbb{Z} \to \Gamma$ so $f^{-1}\Gamma_1 = \{0\}$. But given one such f, the homomorphism $\mathbb{Z} \to \Gamma$ defined by $n \mapsto f(2n)$ also has this property and different from f. This contradicts uniqueness in the definition of universal objects.

Example 1.2.9

Let $(\mathbf{Hot})$ be the category of CW complexes, with arrows being given by homotopy classes of continuous functions. If n is a fixed natural number, then the functor $H^n : (\mathbf{Hot})^{\mathrm{opp}} \longrightarrow (\mathbf{Set})$ that sends CW complex S to the nth cohomology group $H^n(S, \mathbb{Z})$ is representable. This is highly non-trivial, and the space that represents H^n is known as the Eilenberg-MacLane Space, usually denoted by $K(\mathbb{Z}, n)$.

We also have those usual examples coming from algebraic geometry, which we will not elaborate.

The most basic example is $\mathscr{O}^n : (\mathbf{Sch}/S)^{\mathrm{opp}} \longrightarrow (\mathbf{Set})$ defined by sending S-scheme U to $\Gamma(U, \mathscr{O}_U)^n = \mathscr{O}_U(U)^n$. This is represented by $\mathbb{A}^n_S$. On the other hand, the functor $\mathscr{O}^*$ that sends S-scheme U to $\mathscr{O}(U)^*$ is represented by $\mathbb{A}^1_S \backslash \{0\}$. Finally, the functor that sends U to the set of line bundles $\mathscr{L}$ together with $n+1$ sections s_i, mod out by the appropriate equivalence, is represented by $\mathbb{P}^n$.

1.3 Sites

Now we have a more general definition of presheaves, what about sheaves in this kind of generality?

To do this, let's look at vanilla definition of sheaves. We say presheaf $\mathscr{F}$ on a topological space X is a sheaf if, whenever $U \subseteq X$ is open, and $U = \bigcup_i U_i$ is an open cover, if we have $f_i \in \mathscr{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, then there exists unique $f \in \mathcal{F}(U)$ so $f|_{U_i} = f_i$.

This is nicely summarized by saying

$$\mathscr{F}(U) \xrightarrow{\ \rho\ } \prod_i \mathscr{F}(U_i) \overset{\rho_1}{\underset{\rho_2}{\rightrightarrows}} \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

is an equalizer diagram. Here $\rho(f) = (f|_{U_i}) \in \prod_i \mathscr{F}(U_i)$, and $\rho_1(f_i) = (f_i|_{U_i \cap U_j})$ and $\rho_2(f_i) = (f_j|_{U_i \cap U_j})$.

Example 1.3.1

Before we recall the definition of equalizer, we give an instance of the map ρ_1 and ρ_2. Say $U = U_1 \cup U_2$ and $f_i \in \mathscr{F}(U_i)$. Then we get $\prod_{i,j} \mathscr{F}(U_i \cap U_j)$ is the same as

$$\mathscr{F}(U_1 \cap U_1) \times \mathscr{F}(U_1 \cap U_2) \times \mathscr{F}(U_2 \cap U_1) \times \mathscr{F}(U_2 \cap U_2)$$

Thus,

$$\rho_1(f_i) = (f_1|_{U_1 \cap U_1}, f_1|_{U_1 \cap U_2}, f_2|_{U_2 \cap U_1}, f_2|_{U_2 \cap U_2})$$
$$\rho_2(f_i) = (f_1|_{U_1 \cap U_1}, f_2|_{U_1 \cap U_2}, f_1|_{U_2 \cap U_1}, f_2|_{U_2 \cap U_2})$$

Therefore, asking $\rho_1(f_i) = \rho_2(f_i)$ is the same as asking $f_1|_{U_1 \cap U_2} = f_2|_{U_2 \cap U_1}$.

Now, recall $A \xrightarrow{f} B \underset{g}{\overset{h}{\rightrightarrows}} C$ is an equalizer if $gf = hf$ and for all $A' \xrightarrow{f'} B$ such that $gf' = hf'$ there exists unique $\alpha : A' \to A$ such that $f\alpha = f'$. In other word, we have the following diagrams

$$\begin{tikzcd}
A' \arrow[dr, dashed, "\exists! \alpha"'] \arrow[drr, "f'"] & & & \\
& A \arrow[r, "f"'] & B \arrow[r, shift left, "h"] \arrow[r, shift right, "g"'] & C
\end{tikzcd}$$

In other word, in the sheaf axiom equalizer, if we get $(f_i) \in \mathscr{F}(U_i)$ such that $\rho_1(f_i) = \rho_2(f_i)$, then there exists unique $f \in \mathscr{F}(U)$ such that f maps to (f_i). Indeed, to see we get unique f if (f_i) satisfies $\rho_1(f_i) = \rho_2(f_i)$, we just take A' in the above diagram to be $\prod_i \mathscr{F}(U_i)$ and f' be the identity. Then by uniqueness of equalizer diagram, we must have an element $f \in \mathscr{F}(U)$ that does the trick.

Then, Grothendieck's insigh is that, to define sheaves, we don't need the full strength of topology (we don't need unions!). We just need intersections and a notion of when a collection forms a cover (e.g. $U_i \subseteq U$ and $U = \bigcup_i U$).

By doing this, we get what's called Grothendieck topology, which works for any category. Next time we will define Grothendieck topologies and define sheaves.

Above we talked a lot about motivations. Recall if $\mathcal{C}$ is a category, then $\mathrm{Pre}(\mathcal{C}) = \hat{\mathcal{C}}$ denotes the presheaves on $\mathcal{C}$, i.e. they are just functors $F : \mathcal{C}^{\mathrm{opp}} \to (\mathbf{Sets})$.

This is a lot of abstraction, and the presheaves we grow up with are when $\mathcal{C} = \mathrm{Op}(X)$, the category of open sets in X. In particular, we sort of need a topology to define presheaves, but this is false, and Grothendieck realized we only need some weaker notion of abstract coverings.

Recall, $\mathscr{F}$ is a sheaf on a topological space X if whenever $U = \bigcup_i U_i$, we have an equalizer

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

In sets, this is actually pretty concrete, i.e. in the category $(\mathbf{Sets})$, we have

$$A \longrightarrow B \underset{g}{\overset{f}{\rightrightarrows}} C$$

is an equalizer iff $A = \{b \in B : f(b) = g(b)\}$.

So, back to the Grothendieck's insight into topology. In particular, he observed we don't need the full union axiom for topologies, while intersection is still needed.

Definition 1.3.2

Let $\mathcal{C}$ be a category. A ***Grothendieck topology on*** $\mathcal{C}$ is: for every $X \in \mathcal{C}$, a particular subset $\mathrm{Cov}(X) \subseteq \mathrm{PowerSet}(\{Y \to X : Y \in \mathcal{C}\})$, which are called the ***covering of*** X. This $\mathrm{Cov}(X)$ must satisfy:

1. if $V \xrightarrow{\sim} X$, then $\{V \xrightarrow{\sim} X\} \in \mathrm{Cov}(X)$
2. if $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $Y \to X$, then the fiber product $Y \times_X X_i$ exists in $\mathcal{C}$, and $\{Y \times_X X_i \to Y\}_{i\in I} \in \mathrm{Cov}(Y)$.
3. if $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $\{V_{ij} \to X_i\}_{j\in J_i} \in \mathrm{Cov}(X_i)$, then

$$\{V_{ij} \to X\}_{i\in I, j\in J_i} \in \mathrm{Cov}(X)$$

We note, the three conditions correspond to the traditional topological spaces:

1. the first condition means X is a covering of X itself
2. the second condition means the pullback of covering is a covering
3. the third condition means we can refine coverings

We will see this example below in more details.

Definition 1.3.3

A category $\mathcal{C}$ with a choice of Grothendieck topology is called a ***site***.

Example 1.3.4

Let X be a topological space, consider the category $\mathrm{Op}(X)$. Then, we define

$$\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$$

if and only if $\bigcup_{i\in I} U_i = U$. Recall in $\mathrm{Op}(X)$ we have $U_i \to U$ iff $U_i \subseteq U$. This makes $\mathrm{Op}(X)$ an site.

Let's check the three axioms:

1. In our case, $V \xrightarrow{\sim} U$ if and only if $V = U$ (we have double inclusions). This is indeed true, because $U = \bigcup U$.
2. This is a little bit more intersting, but if we actually think about what fibered products are in $\mathrm{Op}(X)$, we would realize it is just intersections. Indeed, suppose we have

$$\begin{tikzcd} & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

then the fibered product is indeed $U \cap V$ as the arrows above are all inclusions. Hence, we see this means, if we have $U = \bigcup_i U_i$ and $V \subseteq U$, then from basic topology we see

$$\{U_i \times_U V \to V\}_{i\in I} = \{U_i \cap V \subseteq V\}_{i\in I}$$

is indeed a cover of V.

3. In our case, this is just $U = \bigcup_i U_i$ and $U_i = \bigcup_j V_{ij}$ then $U = \bigcup_{i,j} V_{ij}$. Hence the third axiom is satisfied.

Definition 1.3.5

If X is a scheme, then $\mathrm{Op}(X)$ with the above Grothendieck topology is called a ***small Zariski site*** of X.

Of course there is also a big Zariski site. First, let $(\mathbf{Sch})$ be the category of schemes, and $(\mathbf{Sch/X})$ be the category of X-schemes.

Example 1.3.6: Big Zariski Site

Let $\mathcal{C} = (\mathbf{Sch/X})$, then we consider the Grothendieck topology obtained by define

$$\{Y_i \to Y\}_{i\in I} \in \mathrm{Cov}(Y \to X)$$

if and only if $Y_i \hookrightarrow Y$ is an open immersion, and $\bigcup_i Y_i = Y$. This is what's called ***big Zariski sites***. We can think of this as, consider all small Zariski sites of Y where $Y \in (\mathbf{Sch/X})$, then sandwich all those small Zariski sites together, we get the big Zariski site of X.

Example 1.3.7

In general, we can localizing a site. If $\mathcal{C}$ is a site, and $X \in \mathcal{C}$, consider $\mathcal{C}_X$ be the category with objects $Y \to X$ and morphisms being

$$\begin{tikzcd} Y' \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd}$$

Then we define the localization on $\mathcal{C}_X$ to be,

$$\left\{ \begin{tikzcd} Y' \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd} \right\} \in \mathrm{Cov}(Y \to X)$$

if and only if $\{Y_i \to Y\}_{i\in I} \in \mathrm{Cov}(Y)$ in $\mathcal{C}$.

In particular, we have big Zariski site of X localized at $Y \to X$ is the big Zariski site of Y.

To make sure we don't lose track, we note the main motivation for this whole business of Grothendieck topology is to get new cohomology theories (e.g. étale cohomology). This is because sheaf cohomology has its problems, e.g. if X is a complex manifold (smooth projective $\mathbb{C}$ scheme), you would like a cohomology theory $H^*(X)$ that recovers topological cohomology. The reason why Grothendieck want to do this is because he wants to prove the Weil conjectures.

Definition 1.3.8

A presheaf $\mathscr{F}$ on a site $\mathcal{C}$ is a ***sheaf***, if for all $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, the sequence

$$\mathscr{F}(U) \longrightarrow \prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathscr{F}(U_i \times_U U_j)$$

is an equalizer (sometimes we say this is "exact sequence").

If you don't like equalizers, the above can be reformulated concretely. This above definition says, if for all $f_i \in \mathscr{F}(U_i)$ such that the image of f_i in $\mathscr{F}(U_i \times_U U_j)$ equal image of f_j in $\mathscr{F}(U_i \times_U U_j)$ for all i, j, then there exists unique $f \in \mathscr{F}(U)$ such that f_i is equal the image of f in $\mathscr{F}(U_i)$ for all i.

Definition 1.3.9

A ***presheaf*** $\mathscr{F}$ on a site $\mathcal{C}$ is called ***separated*** if for all $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, $\mathscr{F}(U) \hookrightarrow \prod_i \mathscr{F}(U_i)$ is an injection.

Theorem 1.3.10

*If $\mathcal{C}$ is a site, then $\mathrm{sh}(\mathcal{C}) := ($**Sheaves on** $\mathcal{C}) \hookrightarrow \mathrm{Pre}(\mathcal{C})$ (by definition this is a full subcategory) has a left adjoint.*

If you are not swimming in the language of category, this can be translated concretely. The above statement says, for all $\mathscr{F} \in \mathrm{Pre}(\mathcal{C})$, there exists $\mathscr{F} \to \mathscr{F}^a$ with $\mathscr{F}^a \in \mathrm{sh}(\mathcal{C})$ such that $\forall \mathscr{G} \in \mathrm{sh}(\mathcal{C})$, we have diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr] & \mathscr{G} \\ & \mathscr{F}^a \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

We call $\mathscr{F}^a$ the ***sheafification*** of $\mathscr{F}$.

Let's first recall the definition of adjoint and see why being left adjoint means the above.

Definition 1.3.11

Let $\mathcal{C}_1, \mathcal{C}_2$ be two categories, $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_1$. Then we say G is the ***right adjoint*** of F and F is the ***left adjoint*** of G, if there exists isomorphism of functors

$$\phi : \mathrm{Hom}_{\mathcal{C}_2}(F(-), -) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_1}(-, G(-))$$

Now, by saying $\mathrm{sh}(\mathcal{C})$ has a left adjoint, we mean there exists $g : \mathrm{Pre}(\mathcal{C}) \to (\mathbf{She}(\mathcal{C}))$ such that we have isomorphism between homs. Here we write $(\mathbf{She}(\mathcal{C}))$ in short for $(\mathbf{Sheaves}(\mathcal{C}))$. Now, given $\mathscr{F}$, we already have $g(\mathscr{F}) =: \mathscr{F}^a$ for free by definition. To get an arrow $\mathscr{F} \to \mathscr{F}^a$, we just note $\mathrm{Hom}(\mathscr{F}, \mathrm{sh}(\mathscr{F}^a)) \cong \mathrm{Hom}(\mathscr{F}^a, \mathscr{F}^a)$ and so we can pullback the identity $\mathscr{F}^a \xrightarrow{\mathrm{Id}} \mathscr{F}^a$ to an arrow $\mathscr{F} \to \mathscr{F}^a$. Next, for any $\mathscr{F} \xrightarrow{f} \mathscr{G} \in \mathrm{Hom}(\mathscr{F}, \mathscr{G})$, we see $\mathrm{Hom}(\mathscr{F}, \mathrm{sh}(\mathscr{G})) \cong \mathrm{Hom}(\mathscr{F}^a, \mathscr{G})$ and thus we get an arrow unique $f^a : \mathscr{F}^a \to \mathscr{G}$. Next, to see those arrows we just got commute, note we get diagram

$$\begin{tikzcd}
\mathscr{F}^a \arrow[r, "f^a"] & \mathscr{G} \\
\mathscr{F}^a \arrow[u, "\mathrm{Id}"] \arrow[ur, "f^a"'] &
\end{tikzcd}$$

But now pullback the $\mathrm{Id} : \mathscr{F}^a \to \mathscr{F}^a$ arrow we get the canonical arrow $\mathscr{F} \to \mathscr{F}^a$ and the whole diagram still commutes (we also need to pullback the upright f^a arrow to $\mathscr{F} \to \mathscr{G}$).

In Hartshorne, the idea is that $\mathscr{F}^a(U)$ is going to be a subset of $\prod_{p \in U} \mathscr{F}_p$ such that consists of "compatible germs". However, in sites, we cannot do this at all, because we don't even have a topology, i.e. for us $U \in \mathcal{C}$, it does not have points, so our proof is going to be complicated.

Proof. We will only give sketch. So, we have

$$\mathrm{sh}(\mathcal{C}) \subseteq (\mathbf{Separated\ presheaves}) \subseteq \mathrm{Pre}(\mathcal{C})$$

and it suffices to show we have each inclusion has a left adjoint. We will only do the left adjoint between separated presheaf and presheaf.

Suppose $\mathscr{F} \in \mathrm{Pre}(\mathcal{C})$ is a presheaf. We want left adjoint $\mathscr{F} \to \mathscr{F}^s$. We define

$$\mathscr{F}^s(U) := \mathscr{F}(U)/\sim$$

where $a, b \in \mathscr{F}(U)$ we say $a \sim b$ if there exists $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ such that a, b map to same thing in $\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i)$.

Why is $\mathscr{F}^s$ a presheaf? If $V \to U$ and $a, b \in \mathscr{F}(U)$ such that $a \sim b$, then there exists a covering $\{U_i \to U\} \in \mathrm{Cov}(U)$ such that in $\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i)$ we have a, b maps to the same image. Then by axiom 2, we know $\{U_i \times_U V \to V\} \in \mathrm{Cov}(V)$ is a covering of V. Then we have a commutative diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d, "\rho"] & \prod_i \mathscr{F}(U_i) \arrow[d] \\
\mathscr{F}(V) \arrow[r] & \prod_i \mathscr{F}(U_i \times_U V)
\end{tikzcd}$$

since a, b maps to the same image in the above vertical arrow, it maps to the same image in $\prod_i \mathscr{F}(U_i \times_U V)$. However, since the diagram is commutative, this means $\rho(a)$ and $\rho(b)$ must map to the same image in the bottom vertical arrow.

Hence we see $\rho(a) \sim \rho(b)$ and so

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}^s(U) \arrow[d, dashed, "\exists!"] \\ \mathscr{F}(V) \arrow[r] & \mathscr{F}^s(V) \end{tikzcd}$$

which shows $\mathscr{F}^s$ is a presheaf. Also, it is clearly separated (by the definition of our equivalent relation). Next it remains to show this is an adjoint, i.e.

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr] & \mathscr{G} \\ & \mathscr{F}^s \arrow[u, "\exists!"'] \end{tikzcd}$$

To that end, just note for each $V \to U$, we get diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "g_U"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \mathscr{F}(V) \arrow[r, "g_V"] & \mathscr{G}(V) \end{tikzcd}$$

we just define the map $\mathscr{F}^s(U) \to \mathscr{G}(U)$ to be $[a] \mapsto g_U(a)$. To see this is well-defined, we just note by the above exact same argument, we get $[a] = [b]$ then $g_U(a) = g_U(b)$ and hence this map $\mathscr{F}^s(U) \to \mathscr{G}(U)$ is well-defined. The functoriality is easy to check.

Next, if $\mathscr{F}$ is separated, we want a sheafification $\mathscr{F}^a$. What we want is that we get the right equalizer. Thus, what we have to have is $\mathscr{F}^a(U)$ must be identified as the elements of $\prod_i \mathscr{F}^a(U_i)$ that gets identified in the product $\prod_{i,j} \mathscr{F}(U_i \times_U U_j)$. But this depends on the covering, hence we define an equivalent relation on all covering.

In particular, we define

$$\mathscr{F}^a(U) := \text{ set of all pairs } \{(\{U_i \to U\}_{i \in I}, \{a_i\}_{i \in I})\} \text{ module } \sim$$

This time, we say

$$(\{U_i \to U\}, \{a_i\}) \sim (\{V_j \to U\}, \{b_j\})$$

iff $a_i|_{U_i \times_U V_j} = b_j|_{U_i \times_U V_j}$ for all i, j.

Thus, we get $\mathscr{F}(U) \to \mathscr{F}^a(U)$ is given by

$$a \mapsto (\{U = U\}, \{a\})$$

Then, we leave it as an exercise to check $\mathscr{F}^a$ is a sheaf and the map is left adjoint.

Remark 1.3.12

All of this works for sheaves/presheaves valued in groups, modules, rings, etc.

1.4 Étale Maps

Let's stop talk about categories and do some geometry (of étale morphisms).

So, the idea is that étale morphisms are "covering spaces" (from algebraic topology). There are tons of characterizations of étale morphisms, and we will talk about five of them.

Before we give formal definitions, let's see one instance where étale maps pops up. Recall from classical topology theory, for a "nice" topological space X, we get the following correspondence

$$\{\text{finite covering } f : X \to Y\} \longleftrightarrow \{\text{finite } \pi(X,x)\text{-sets}\}$$

But then when we move to the realm of schemes, things become disastrous. That is, we have the following example.

Example 1.4.1

Consider $g(U,V) = V^3 + 2V^2 - 15V - 4U$ and $C = \{(u,v) \in \mathbb{C}^2 : g(u,v) = 0\}$ endowed with classical Euclidean topology. Then we see projection $f : C \to \mathbb{C}$ via $f(u,v) = u$ is almost a finite cover. That is, $\# f^{-1}(u)$ is always equal 3 if $u \notin \{-\frac{100}{27}, 9\}$, as oen can compute the discriminant of quadratic polynomial $g(u,V)$. Therefore, $f : C \to \mathbb{C}\backslash\{-\frac{100}{27}, 9\}$ is a finite covering of topological spaces and the degree is 3.

Now turn this into schemes, we define

$$A = \mathbb{C}[U, \frac{1}{(27U+100)(U-9)}], \quad B = A[V]/gA[V]$$

and so we get $f : \operatorname{Spec} B \to \operatorname{Spec} A$. This is not a trivial covering at all! Indeed, take a look at the generic point $\xi \in \operatorname{Spec} A$, we see its local ring is $\operatorname{Frac}(A) = \mathbb{C}(U)$ and the fiber of f over ξ is $\operatorname{Spec}(\operatorname{Frac}(B))$. However, $\operatorname{Frac}(B)/\operatorname{Frac}(A)$ is a degree three field extension, i.e. locally around ξ our map $f : \operatorname{Spec} B \to \operatorname{Spec} A$ can never be trivial and hence can never be a covering space.

What is the fix here? We change our topology from Zariski topology to étale topology, and we consider finite étale maps. In this case, $f : \operatorname{Spec} B \to \operatorname{Spec} A$ would turn out to be a covering, as desired.

In particular, we have a beautiful analogy to the topological fundamental group, i.e. finite étale coverings of X correspond bijectively to finite $\pi_1^{\text{ét}}(X,x)$-sets. Here $\pi_1^{\text{ét}}(X,x)$ is a profinite group, hence it has a topology on it, and we requrie the $\pi_1^{\text{ét}}(X,x)$-action on the sets to be continuous.

Thus, we see from topological analogy, it is necessary for us to step out from the world of Zariski topology and consider finer structures. Now let's jump into formal definitions.

Definition 1.4.2

A morphism $f: Y \to X$ is ***quasi-finite*** if it is of finite type and for all $x \in X$, the fiber $Y_x := Y \times_X \operatorname{Spec} \kappa(x)$ is a finite set.

Definition 1.4.3

A morphism $f: X \to Y$ is ***locally quasi-finite*** if for all $y \in Y$, there exists open neighbourhood $y \in U$ such that $f(U) \subseteq V$ and $f|_U : U \to V$ is quasi-finite (equivalently, $f|_U : U \to X$ is quasi-finite).

Example 1.4.4

Consider $\coprod_{i=1}^{\infty} \operatorname{Spec} K \to \operatorname{Spec} K$ is locally quasi-finite but not quasi-finite. This is because if we take the inverse image of the point in $\operatorname{Spec} K$, you get infinitely many elements in the fiber, but locally this is finite.

Definition 1.4.5

A morphism $f: Y \to X$ is ***étale*** if f is smooth and locally quasi-finite (i.e. étale is smooth plus relative dimension 0).

Let's consider the key example that we will use through out the course.

Example 1.4.6

Let K be a field, then $Y \to \operatorname{Spec} K$ is étale iff $Y = \coprod_i \operatorname{Spec} L_i$ where L_i/K is finite separable extension of fields.

Proposition 1.4.7

1. *Compostion of étale morphisms are étale, i.e. $f: Z \to Y, g: Y \to X$ are both étale, then $g \circ f$ is étale.*
2. *If $f: Y \to X$ is étale, then the base change is étale, i.e. if we have*

$$\begin{tikzcd} Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r, "\acute{e}t"] & X \end{tikzcd}$$

   then $Y \times_X Z \to Z$ is étale.
3. *If*

$$\begin{tikzcd} Z \arrow[r] \arrow[dr, "\acute{e}t"'] & Y \arrow[d, "\acute{e}t"] \\ & X \end{tikzcd}$$

then $Z \to Y$ is étale.

Example 1.4.8

We consider the ***small étale site*** of X to be the following category: objects being $Y \to X$ with arrows being etale, and morphisms are etale triangles

$$\begin{tikzcd} Y' \arrow[r, "\text{ét}"] \arrow[dr, "\text{ét}"'] & Y \arrow[d, "\text{ét}"] \\ & X \end{tikzcd}$$

Then, the Grothendieck topology would be, $\{Y_i \xrightarrow{et} Y\} \in \mathrm{Cov}(Y \xrightarrow{et} X)$ iff

$$\coprod_i Y_i \twoheadrightarrow Y$$

is surjective.

Example 1.4.9

If $\mathscr{F}$ is a sheaf on the small étale site of $\operatorname{Spec} K$, K a field. Then, suppose L/K is Galois extension with Galois group G. Then

$$\mathscr{F}(\operatorname{Spec} K) = \mathscr{F}(\operatorname{Spec} L \xrightarrow{et} \operatorname{Spec} K)^G$$

which is the G-invariant of $\mathscr{F}(\operatorname{Spec} L)$. Here that $\mathscr{F}(\operatorname{Spec} L \xrightarrow{et} \operatorname{Spec} K)^G$ is just functors on the objects.

Example 1.4.10

Suppose $f : \mathbb{A}^1 \to \mathbb{A}^1$ is given by $x \mapsto x^2$. We claim this is not étale. This is clearly quasi-finite, thus we need to show it is not smooth. Consider the pullback F_0 where $\operatorname{Spec} k \to \mathbb{A}^1$ is the point 0,

$$\begin{tikzcd} F_0 \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

Let's compute what the fibered product is: geometrically we are just looking at the map $\operatorname{Spec} k[x,y]/(x-y^2) \to \operatorname{Spec} k[x]$ and $\operatorname{Spec} k \to \mathbb{A}^1$ correspond to the point (x), and so we are looking for the pushout for the following diagram

$$\begin{tikzcd} ? & k[x,y]/(x-y^2) \arrow[l] \\ k[x]/(x) \arrow[u] & k[x] \arrow[l, "x \mapsto x"'] \arrow[u, "x \mapsto x"'] \end{tikzcd}$$

But then we see we have a formula for this kind of situation, i.e. the pushout is precisely

$$k[x,y]/(x-y^2) \otimes_{k[x]} k[x]/(x)$$

But recall in general, for ring R, ideal I and R-module M, we get $R/I \otimes_R M \cong_R M/IM$ as modules. Thus we get

$$k[x,y]/(x-y^2) \otimes_{k[x]} k[x]/(x) \cong (k[x,y]/(x-y^2))/(x) \cong k[y]/(y^2)$$

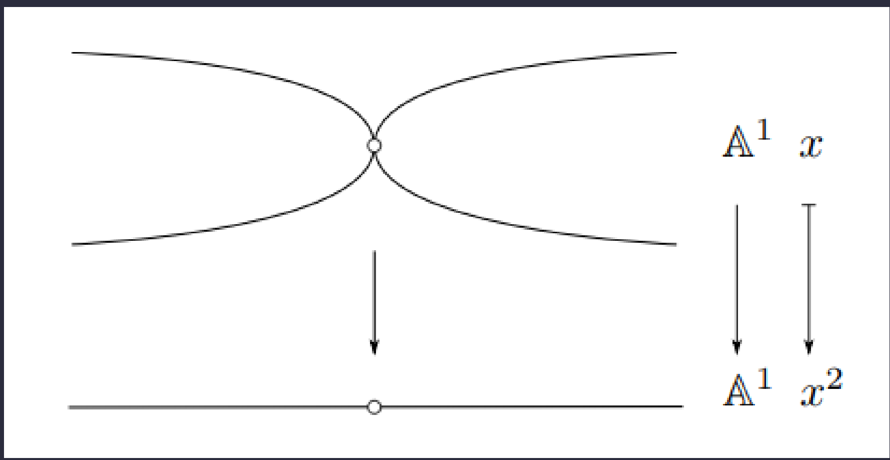
Now convince yourself this is also a map of $k[x]$-algebras.

Thus we get

$$F_0 = \operatorname{Spec} k[x]/(x^2)$$

But then F_0 is the dual number and hence it is not a field, which implies we get a map $F_0 \to \operatorname{Spec} k$ that's not étale. Hence the original map $f : \mathbb{A}^1 \to \mathbb{A}^1$ is not étale.

The problem of our map f is that it is ramified at the origin, i.e. we are sandwich the parabola to a single line, and hence the origin is squeezed twice:



Put it in the other perspective, we see $\frac{d}{dx}[x^2] = 2x$ and hence it vanishes at 0. In other word, the module of differential

$$\Omega^1_{\mathbb{A}^1/\mathbb{A}^1} = \frac{k[x]dx}{d(x^2)} = \frac{k[x]dx}{2xdx} \cong \frac{k[x]}{x}$$

Viz, this is a $k[x]$-module supported at 0, i.e. if we specialize x to be non-zero, the module is 0, but if $x \mapsto 0$, the module is non-zero.

The above example actually give an different way to define what étale maps are.

Definition 1.4.11

If $f : Y \to X$ is locally of finite presentation, then f is ***unramified*** if $\Omega^1_{Y/X} = 0$.

Proposition 1.4.12

Let f be locally of finite presentation. Then $f : Y \to X$ is étale if and only if f is smooth and unramified iff f is smooth and $\Omega^1_{Y/X} = 0$.

Note almost all maps in this course will be locally of finite presentation. Hence we will just drop this assumption.

Proposition 1.4.13

The map $f : Y \to X$ is étale iff f locally of finite presentation and for all $y \in Y$, there exists open neighbourhood $y \in U$ and $f(y) \in V$, such that $f(U) \subseteq V$ and $f|_U : U \to V$ is "standard étale", i.e. $V = \operatorname{Spec} R$ and $U = \operatorname{Spec}(R[x]/f)_g$ for some g, where $f \in R[x]$ and $\frac{d}{dx} f$ is a unit in $(R[x]/f)_g$.

Definition 1.4.14

We say f is ***formally smooth***, if

$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r] \arrow[d, hook] & Y \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists"] & X \end{tikzcd}$$

for all A and I, where $I \subseteq A$ is an ideal with $I^2 = 0$. We say f is ***formally étale*** if it is formally smooth and the dotted arrow is unique, i.e. we have $\exists!$.

The way to think about this is that, we would have (where $A = k[x]/(x^2)$ and $I = (x)$)

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ \operatorname{Spec} k[x]/(x^2) \arrow[r] & X \end{tikzcd}$$

and hence the dotted arrow is just choosing a tangent vector at y, i.e. formal smoothness is every $y \in Y$ can be extended to a tangent vector at y.

The following characterization is the usual definition of étale.

Proposition 1.4.15

Let $f : Y \to X$, then f is étale (resp. smooth) iff it is formally étale (resp. formally smooth) and locally of finite presentation.

Proposition 1.4.16

Let $f : Y \to X$, then f is étale iff f is flat and unramified.

Example 1.4.17

Recall we defined the small étale site of X to be as follows. The objects are $Y \xrightarrow{et}$

X, and morphisms are étale triangles. THen the coverings are, $\{Y_i \to Y\}_{\in I} \in \mathrm{Cov}(Y \to X)$ where $\coprod Y_i \twoheadrightarrow Y$.

Based on this, we can define the ***big étale site*** of X, where now $\mathcal{C} = (\mathbf{Sch})/X$, and the coverings are given by

$$\{Y_i \to Y\} \in \mathrm{Cov}(Y \to X)$$

if $Y_i \xrightarrow{et} Y$ and $\coprod Y_i \twoheadrightarrow Y$.

Example 1.4.18

Let $\mathscr{F}$ be a sheaf on small étale site of $\operatorname{Spec} K$, K a field. Let L/K be (finite) Galois field extension with group G. Then, suppose we are given the singleton $\{\operatorname{Spec} L \to \operatorname{Spec} K\}$, then this is in $\mathrm{Cov}(\operatorname{Spec} K)$.

Now we know G acts on L over K, thus for all $g \in G$, we get

$$\begin{tikzcd} L \arrow[r, "g", "\cong"'] & L \\ & K \arrow[ul, hook] \arrow[u, hook] \end{tikzcd}$$

Thus we obtain a map

$$\begin{tikzcd} \operatorname{Spec} L \arrow[r, "g"] \arrow[dr] & \operatorname{Spec} L \arrow[d] \\ & \operatorname{Spec} K \end{tikzcd}$$

This is a self map of $\operatorname{Spec} L \to \operatorname{Spec} K$ in the étale site. So since $\mathscr{F}$ is a presheaf, we get $\mathscr{F}(L) \xrightarrow{g^*} \mathscr{F}(L)$. Thus G acts on $\mathscr{F}(L)$.

The fact $\mathscr{F}$ is a sheaf implies

$$\mathscr{F}(K) \longrightarrow \mathscr{F}(L) \rightrightarrows \mathscr{F}(L \otimes_K L)$$

is exact. To know what's going on, we compute $L \otimes_K L$. But first, note L/K is separable, we get $L = K(\alpha)$ for some $\alpha \in L$. The minimal polynomial of α is exactly $f(x) = \prod_{g \in G}(x - g(\alpha)) \in K[x]$. Hence we see

$$\begin{aligned} L \otimes_K L &= L \otimes_K K(\alpha) \\ &= L \otimes_K K(x)/f(x) \\ &= L[x]/f(x) \\ &\cong \bigoplus_{g \in G} L(x)/(x - g(\alpha)) \\ &\cong \bigoplus_{g \in G} L \end{aligned}$$

It turns out, the two maps from L to $L \otimes_K L \cong \bigoplus_{g \in G} L$ are given by, $\beta \mapsto (\beta)_{g \in G}$ and $\beta \mapsto (g(\beta))_{g \in G}$.

Then, since $\mathscr{F}$ is a sheaf, we get

$$\mathscr{F}(K) = \{\gamma \in \mathscr{F}(L) : \text{ under the two maps } \gamma \text{ has same image}\}$$

But $\gamma \in \mathscr{F}(L)$ has the same image under the two maps iff $(\gamma)_{g\in G} = (g(\gamma))_{g\in G}$. Hence

$$\mathscr{F}(K) = \{\gamma \in \mathscr{F}(L) : \forall g, g^*(\gamma) = \gamma\} = \mathscr{F}(L)^G$$

1.5 Fppf Sites

Definition 1.5.1

A ***topos*** is a category $\mathcal{C}$ equivalent to the category of sheaves on a site.

Definition 1.5.2

A ***morphism of topoi*** $f : T \to T'$ is a triple $f := (f^*, f_*, \phi)$, where

$$f_* : T' \to T$$

$$f^* : T \to T'$$

and these are an adjoint pair with f^* equal the left adjoint of f_*, and

$$\phi : \mathrm{Hom}_T(f^*(-), -) \xrightarrow{\sim} \mathrm{Hom}_{T'}(-, f_*(-))$$

is a choice of isomorphism that's natural in the two dashes such that f^* commutes with finite limits (note by definition of adjoint we have the two sets being isomorphic, and we just make a particular choice of ϕ).

We will not give a full definition of what is the meaning of finite limit, but f^* commutes with finite limits is equivalent to saying that,

$$f^*(\mathscr{F}_1 \times \dots \times \mathscr{F}_n) = f^*\mathscr{F}_1 \times \dots \times f^*\mathscr{F}_n$$

and if $\mathscr{F} \to \mathscr{G} \rightrightarrows \mathscr{H}$ is equalizer, then $f^*\mathscr{F} \to f^*\mathscr{G} \rightrightarrows f^*\mathscr{H}$.

The full definition of finite limit is just limits over finite diagrams (hence we don't take infinite diagrams).

For example, if we have the following diagram (the diagram is the ones with the

A_i)

$$
\begin{tikzcd}
& B \arrow[ddl, dashed] \arrow[ddd, dashed] \arrow[ddr, dashed] \arrow[drr, dashed] \arrow[ddrrr, dashed, bend left] & & & \\
& & & A_4 \arrow[dl] \arrow[dr] & \\
A_1 \arrow[dr, shift left] \arrow[dr, shift right] & & A_3 \arrow[dl] & & A_5 \\
& A_2 & & &
\end{tikzcd}
$$

Then we say B is the limit if there exists unique dotted arrows making the above diagram commutes, and if C also has dotted arrows, then C factors through B.

So why is that we only need to check products and equalizer?

Well, because we have another way to construct B: We isolate the source and target (so we isolate all objects that has arrows going out, and all objects that receives arrows), and make product between those, i.e. we have

$$A_1 \times A_3 \times A_4 \rightrightarrows A_2 \times A_5$$

then the limit B is just the equalizer of the above diagram, i.e.

$$B = \mathrm{Eq}\left(A_1 \times A_3 \times A_4 \rightrightarrows A_2 \times A_5 \right)$$

Definition 1.5.3

A ***continuous map*** of sites $f : \mathcal{C} \to \mathcal{C}'$ is a functor such that for all $X \in \mathcal{C}$, for all $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ we have $\{f(X_i) \to f(X)\}_{i \in I} \in \mathrm{Cov}(f(X))$, and f commutes with finite fiber product, if they exists.

Example 1.5.4

Suppose $f : X \to Y$ be a map between topological spaces. Then we get continuous map of sites $f^{-1} : \mathrm{Op}(Y) \to \mathrm{Op}(X)$. Indeed, $U \subseteq Y$ maps to $f^{-1}(U)$ and hence it is continuous.

From what follows, we will show $f : \mathcal{C} \to \mathcal{C}'$ is continuous then there exists $f_* : T \to T'$. Once we get a map $f_* : T \to T'$, we automatically get f^* for free by the following proposition. However, we will not prove this result.

Proposition 1.5.5

If $f : \mathcal{C}' \to \mathcal{C}$ is continuous map of sites, then $f_* : T' \to T$ has left adjoint $f^* : T \to T'$. If $\mathcal{C}'$ has finite limit and f commutes with finite limits, then f^* commutes with finite limits, i.e. (f^*, f_*) yields $T' \to T$ a map of topoi.

Example 1.5.6

If $f : X \to Y$ is a map of schemes. Then we get a map from the small étale site of Y to the small étale site of X given by

$$(Z \xrightarrow{et} Y) \mapsto (Z \times_X Y \xrightarrow{et} X)$$

This is a continuous map of sites by properties of fibered product/pullback.

As a result (by the theorem below), we get a map of topoi $X_{et} \to Y_{et}$.

Example 1.5.7

The above example is not a single case. We can also use pullback to define continuous maps from small Zariski site on X to the small étale site of X, by the map $U \subseteq X \mapsto U \subseteq X$. In particular, we get a map of topoi $X_{et} \to X_{Zar}$

From above we stated a proposition about how continuous maps induce map between topoi.

Let's now define pushforward on topoi. Given $f : \mathcal{C}' \to \mathcal{C}$ continuous map of sites. We get

$$f_* : T \to T'$$

on topoi by

$$(f_*\mathscr{F})(X') = \mathscr{F}(f(X'))$$

We note unlike pushforward of sheaves on schemes, where we define $f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U))$, we used $f(X')$. This is because f itself in our context should be thought as an "inverse" (think of the topological space example).

So why is $f_*\mathscr{F}$ a sheaf?

If $\{X_i' \to X'\}$ is a cover of X', then we want to check the following sequence is exact

$$(f_*\mathscr{F})(X') \longrightarrow \prod_{i \in I} f_*\mathscr{F}(X_i') \rightrightarrows \prod_{i,j} f_*\mathscr{F}(X_i' \times_X X_j')$$

However, by definition, the above sequence is the same as

$$\mathscr{F}(f(X')) \longrightarrow \prod_{i \in I} \mathscr{F}(f(X_i')) \rightrightarrows \prod_{i,j} \mathscr{F}(f(X_i' \times_X X_j'))$$

but f commutes with finite fiber product, and hence

$$\mathscr{F}(f(X_i' \times_X X_j')) = \mathscr{F}(f(X_i') \times_{f(X)} f(X_j'))$$

However, now we see the whole sequence is exact because $\{f(X_i') \to f(X')\}$ is a covering of $f(X')$.

Theorem 1.5.8

If $f : \mathcal{C}' \to \mathcal{C}$ is continuous map of sites. Then $f_* : T \to T'$ has a left adjoint

$f^*: T' \to T$. Moreover, if $\mathcal{C}'$ has finite limits and f commutes with finite limits, then f^* commutes with finite limits. In particular, (f^*, f_*) is a map of topoi $T \to T'$.

We will not prove this, but will mention how to construct f^*.

We just need to construct a left adjoint of f_*. Note f_* also gives a map $\mathrm{Pre}(\mathcal{C}) \to \mathrm{Pre}(\mathcal{C}')$, thus if we can construct a adjoint of $f_*: \mathrm{Pre}(\mathcal{C}) \to \mathrm{Pre}(\mathcal{C}')$, because if f^*_{Pre} is the left adjoint then take f^* be the sheafification $(f^*_{\mathrm{Pre}})^a$ (we composed two left adjoints, so the whole thing is left adjoint).

The construction of f^* is also similar to what we do in baby algebraic geometry.

In particular, we take

$$(f^*\mathscr{F})(U) = \varinjlim_{U'} \mathscr{F}(U')$$

where the colimit is taken over a diagram with objects (U', ρ) where $\rho: U \to f(U')$

Remark 1.5.9

If $X' \in \mathcal{C}'$, then we get representable functor $h_{X'} \in \mathrm{Pre}(\mathcal{C}')$. Thus we get $h^a_{X'} \in T'$ is a sheaf in the topos. Thus we see $f^*(h^a_{X'}) = h^a_{f(X')}$ if $f: \mathcal{C}' \to \mathcal{C}$ is a continuous map of sites.

Let's prove this, as this is just unravel definitions. Note

$$\mathrm{Hom}_T(f^*(h^a_{X'}), \mathscr{F}) = \mathrm{Hom}_{T'}(h^a_{X'}, f_*\mathscr{F}) = \mathrm{Hom}_{\mathrm{Pre}(\mathcal{C}')}(h_{X'}, f_*\mathscr{F})$$

where on the first equality we used f_* and f^* are adjoints, and the second equality is because sheafification is adjoint. Next, by Yoneda, we get

$$\mathrm{Hom}_{\mathrm{Pre}(\mathcal{C}')}(h_{X'}, f_*\mathscr{F}) = (f_*\mathscr{F})(X')$$

However, $(f_*\mathscr{F})(X') = \mathscr{F}(f(X'))$ and Yoneda again we get

$$\mathscr{F}(f(X')) = \mathrm{Hom}_{\mathrm{Pre}(\mathcal{C})}(h_{f(X')}, \mathscr{F})$$

Now use the same trick, we observe

$$\mathrm{Hom}_{\mathrm{Pre}(\mathcal{C})}(h_{f(X')}, \mathscr{F}) = \mathrm{Hom}_T(h^a_{f(X')}, \mathscr{F})$$

and hence we get $h^a_{f(X')} = f^*(h^a_{X'})$.

This is about enough category for today, but before geometry, we need to define one more site, the fppf site.

Definition 1.5.10

We say $A \to B$ is ***of finite presentation*** if B is a finitely generated A-algebra and

$$A^m \to A^n \to B \to 0$$

is exact with n,m finite, i.e. the kernel is finitely generated.

We also have another formulation. Note if we assume B is f.g. A-algebra, then we get surjection $\pi : A[x_1,...,x_n] \twoheadrightarrow B$. Then we say $A \to B$ is of finite presentation if $\ker(\pi)$ is f.g. ideal of $A[x_1,...,x_n]$. In other word, we should think of finite presentation as being finitely generated A-algebra with finitely many relations (i.e. B is f.g. A-algebra means finite many generators, and B is of finite presentation means B has finite number of relations on the finite number of generators).

We can generalize this to schemes.

Definition 1.5.11

We say $f : X \to Y$ is ***locally of finite presentation*** if for all $y \in Y$, there exists open neighbourhood $U = \operatorname{Spec} A \subseteq Y$ of y and $f^{-1}(U)$ has an open affine cover $\bigcup V_i = \bigcup \operatorname{Spec} B_i$ such that $A \to B_i$ is of finite presentation.

Definition 1.5.12

$f : X \to Y$ is ***faithfully flat*** if f is flat and surjective.

Definition 1.5.13

$f : X \to Y$ is ***fppf*** if f is faithfully flat and locally of finite presentation.

Example 1.5.14

If X is a scheme, the ***fppf site*** has category $(\mathbf{Sch/X})$ and coverings

$$\{Y_i \to Y\} \in \operatorname{Cov}(Y)$$

iff each $Y_i \to Y$ is flat and locally of finite presentation, and together we have surjection

$$\coprod_i Y_i \twoheadrightarrow Y$$

The fppf sites play a very very important role in algebraic geometry. One reason is that we have what's called faithfully flat descent.

The idea is that, let P be a property of morphisms of schemes. Then we get commutative diagram

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, two heads, "\text{fppf}"] & X \end{tikzcd}$$

Then, frequently, f has P iff f' has P. This is what's called faithfully flat descent.

We are familiar with the fact that if $X = \bigcup U_i$ is open covering, then $f : Y \to X$ has property P iff $f|_{f^{-1}(U)}$ has property P for all i. This is in fact an example of faithfully flat descent (in the Zariski topology).

Remark 1.5.15

We note Zariski cover is a subset of étale cover and étale cover is a subset of fppf cover. Hence because faithfully flat descent on fppf covers, then it holds for étale and Zariski as well.

Remark 1.5.16

We get maps on topoi:

$$X_{\text{fppf}} \to X_{\text{ét}} \to X_{\text{Zar}}$$

1.6 Faithfully Flat Descent

We will want to work up our way to faithfully flat descent, and the first thing we do is give tons of characterizations.

Definition 1.6.1

A module M is:

1. ***flat*** if $-\otimes_R M$ is exact functor.
2. ***faithfully flat*** if $0 \to N' \to N \to N'' \to 0$ is exact iff $0 \to N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M \to 0$ is exact.

Proposition 1.6.2

The following are equivalent:

1. *M is flat and we have injection $\mathrm{Hom}_R(N, N') \hookrightarrow \mathrm{Hom}(N \otimes M, N' \otimes M)$ for all N, N'.*
2. *M is flat and for all N', we have injection $N' \hookrightarrow \mathrm{Hom}(M, N' \otimes M)$ by $y \mapsto (y \mapsto y \otimes m)$.*
3. *M is faithfully flat.*
4. *$N' \to N$ is injection iff $N' \otimes M \to N \otimes M$ is injection.*
5. *M is flat and $N \otimes M = 0$ then $N = 0$.*
6. *M is flat and for all maximal ideal $\mathfrak{m} \subseteq R$, $M/\mathfrak{m}M \neq 0$.*
7. *M is flat and for all prime ideal $\mathfrak{p} \subseteq R$, $M/\mathfrak{p}M \neq 0$.*

Proof. (1) $\Longleftrightarrow$ (2): If $F \twoheadrightarrow N$, then $F \otimes M \twoheadrightarrow N \otimes M$. Thus we get

$$\begin{array}{ccc} \mathrm{Hom}(N, N'), & \xrightarrow{\phi_{N,N'}} & \mathrm{Hom}(N \otimes M, N' \otimes M) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}(F, N') & \xrightarrow{\phi_{F,N'}} & \mathrm{Hom}(F \otimes M, N' \otimes M) \end{array}$$

where we have injections because $F \otimes M \twoheadrightarrow N \otimes M$ is surjection. We always can take $F = R^I$ to be a free module, thus $\phi_{N,N'}$ is injection iff $\phi_{F,N'}$ is injection[1], i.e. (1) is equivalent to checking (1) when $N = R^I$ is free.

However, $\phi_{N,N'} : \mathrm{Hom}(R^I, N') \to \mathrm{Hom}(R^I \otimes M, R^I \otimes N')$ is just

$$\phi_{N,N'} = \prod_{i \in I} \mathrm{Hom}(R, N') \to \prod_{i \in I} \mathrm{Hom}(M, R^I \otimes N')$$

But if we think what is this map doing, we can assume $|I| = 1$ as all factors are preserved in products. Thus we have

$$N' \xrightarrow{\sim} \mathrm{Hom}(R, N') \to \mathrm{Hom}(M, M \otimes N')$$

and this is exactly the map

$$y \mapsto (1 \mapsto y) \mapsto (m \mapsto m \otimes y)$$

This concludes (1) holds iff (2) holds.

Let's go from (2) to (4). If M is flat, then $N \hookrightarrow N'$ implies $N \otimes M \hookrightarrow M \otimes N'$. So, we have to show the converse, i.e. if $N \to N'$ and $N \otimes M \hookrightarrow N' \otimes M$ then $N \hookrightarrow N'$.

We see

$$\begin{tikzcd} N \arrow[r, hook, "\text{by (2)}"] \arrow[d] & \mathrm{Hom}(M, N \otimes M) \arrow[d] \\ N' \arrow[r, hook, "\text{by (2)}"] & \mathrm{Hom}(M, N' \otimes M) \end{tikzcd}$$

but the right vertical arrow is injective since $N \otimes M \hookrightarrow N' \otimes M$ and $\mathrm{Hom}(M, -)$ is left-exact. This forces $N \to N'$ to be injective as the diagram commutes.

(4) $\to$ (5): Suppose $N \otimes M = 0$, then tensor the map $N \to 0$ by M, we get $N \otimes M = 0 \to 0$ is injective. Thus $N \to 0$ is injective and hence $N = 0$ as desired.

(5) $\Rightarrow$ (7): Let $N = R/\mathfrak{p} \neq 0$ if $\mathfrak{p}$ is prime. Thus $N \otimes M \neq 0$ where $N \otimes M = M/\mathfrak{p}M$.

(7) $\Rightarrow$ (6): Maximal ideals are prime.

(5) $\Leftarrow$ (3): If $N \otimes M = 0$, then consider the sequence $(0 \to N \to 0)$ and tensor with M we get exact sequence $0 \to 0 \to 0$, hence the original sequence $0 \to N \to 0$ must be exact, hence $N = 0$ as desired.

(5) $\Rightarrow$ (3): We have $N' \xrightarrow{\alpha} N \xrightarrow{\beta} N''$ and exact sequence

$$N' \otimes M \xrightarrow{\alpha'} N \otimes M \xrightarrow{\beta'} N'' \otimes M$$

Let H be the cohomology

$$H := \frac{\ker(\beta)}{\mathrm{Im}(\alpha)}$$

[1]Well, to see why this is true, clearly if the bottom arrow is injective then the top arrow must be. On the other hand, assume top arrow is injective. Now pick any $x, y : R^I \to N'$, which is the same as $(x_i), (y_i) \in (N')^I$, but then we see we get $(x_i/K), (y_i/K) \in \mathrm{Hom}(N, N')$. Now use injectivity on the top to conclude the bottom is injective.

We want $H = 0$. However, since M is flat, we have

$$H \otimes M = \frac{\ker \beta'}{\operatorname{Im} \alpha'} = 0$$

and hence $H = 0$ as desired.

It remains to show $(6) \Rightarrow (2)$. We do this by contradiction. Suppose there exists N with $N \to \operatorname{Hom}(M, N \otimes M)$ by $y \mapsto (m \mapsto y \otimes m)$ is not injective, i.e. $\exists x \in N$ such that $x \otimes m = 0$ for all $m \in M$. We will produce a maximal ideal with the quotient equal 0.

Let $L \subseteq N$ be the submodule generated by x, i.e. $L = Rx$. Then we let $I = \ker(R \to L)$ where the map $R \to L$ is given by $1 \mapsto x$. Thus we see $L = R/I$. We will show we can replace N by L.

Note $L \hookrightarrow N$, and since M is flat, we see $L \otimes M \hookrightarrow N \otimes M$. We see $x \otimes m = 0$ when x is viewed as element of N. Hence $x \otimes m = 0$ when x is viewed as element of L. But x generates L, hence $L \otimes M = 0$. Thus we see $0 = L \otimes M = R/I \otimes M = M/IM$. If $\mathfrak{m} \supseteq I$ is the maximal ideal, then $M/\mathfrak{m}M = 0$, which is a contradiction.

Corollary 1.6.2.1

$\operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A$ *is faithfully flat iff B is a faithfully flat A-module.*

Proof. By definition, f is flat iff B is flat A-module. So, f is flat and surjective iff B is flat and if we do base change at closed point $\operatorname{Spec} A/\mathfrak{p} \to \operatorname{Spec} A$, we get non-empty the fibered product is non-empty, i.e. we have

$$\begin{tikzcd} \neq \emptyset \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} A/\mathfrak{m} \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

But this is condition (6) and we are done.

Theorem 1.6.3

If $f : X \to Y$ is flat and locally of finite presentation, then f is open, i.e. $f(U)$ is open if U is open.

Proof. We are going to prove this, but we need to import a block box theorem, i.e. the Chevalley's theorem.

Theorem 1.6.4: Chevalley

If $f : X \to Y$ is locally of finite presentation, then $f(U)$ is constructible if U is constructible. In particular, $f(U)$ is constructible if U is open.

All we need from Chevalley's theorem is that $f(U)$ is constructible when U is open.

We also need another fact from topology:

Fact

Let $E \subseteq Y$ be a constructible set and E is stable under generalization, i.e. $y \in \overline{\{y'\}}$ and $y \in E$ then $y' \in E$. Then E is open.

As a corollary of the above facts, we see if $f : X \to Y$ locally of finite presentation and $U \subseteq X$ open and $f(U)$ is stable under generalization, then $f(U)$ is open. Now let's prove the theorem.

Let $U \subseteq X$ be open, we just need to show $f(U)$ is stable under generalization. We can reduce this to the local case and assume $f : \operatorname{Spec} B \to \operatorname{Spec} A$. We have $y \in f(U)$ and $y \in \overline{\{y'\}}$. Thus, say $f(x) = y$. We want to find x' specialize to x such that $f(x') = y'$.

Thus, we have

$$\begin{tikzcd} x \arrow[d] & x' \arrow[l, dotted] \arrow[d, dotted] \\ y & y' \arrow[l] \end{tikzcd}$$

where the dotted arrows are what we wanted. In particular, this means we get $\mathfrak{q}$ in B that's lies above $\mathfrak{p}$ that correspond to y, and we get $\mathfrak{p}' \subseteq \mathfrak{p}$ that correspond to y'. We want $\mathfrak{q} \supseteq \mathfrak{q}'$ such that $\mathfrak{q}'$ lies over $\mathfrak{p}'$. This is known as "going down".

In other word, after localizing, we get

$$\begin{tikzcd} B_{\mathfrak{q}} & \\ A_{\mathfrak{p}} \arrow[u, "\text{flat}", "\phi"'] & \mathfrak{p}'A_{\mathfrak{p}} \arrow[l, "\supseteq"'] \end{tikzcd}$$

and we want a prime of $B_{\mathfrak{q}}$ lying over $\mathfrak{p}'A_{\mathfrak{p}}$. This is the same as

$$\begin{tikzcd} \operatorname{Spec} B_{\mathfrak{q}} \arrow[d, "g"] \\ \operatorname{Spec} A_{\mathfrak{p}} \end{tikzcd}$$

and we want $\mathfrak{p}' \in \operatorname{Im}(g)$, i.e. we want g to be surjective.

However, g is flat and $\operatorname{Spec} A_{\mathfrak{p}}$ has only one closed point and g surjects on that closed point. Thus, by the equivalence of (6) and (7) of the proposition we proved,

g is surjective on all points. To see this, note $f : \operatorname{Spec} B_{\mathfrak{q}} \to \operatorname{Spec} A_{\mathfrak{p}}$ is surjective on closed points, we see we get the following diagram

$$\begin{tikzcd}
g^{-1}(\mathfrak{p}) \arrow[r] \arrow[d] & \operatorname{Spec} B_{\mathfrak{q}} \arrow[d] \\
\operatorname{Spec} A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \arrow[r] & \operatorname{Spec} A_{\mathfrak{p}}
\end{tikzcd}$$

where we know the fiber of the point $\operatorname{Spec} A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is not empty, i.e. the ring $B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p})$ is not the zero ring. But $B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \cong B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$, which tells us we get (6) to hold, and hence we actually get surjection on all points.

Corollary 1.6.4.1

If $f : X \to Y$ is fppf, and $U \subseteq Y$ is open and quasi-compact (e.g. affine). Then there exists open cover $f^{-1}(U) = \bigcup_j V_j$ with V_j are quasi-compact and $f(V_j) = U$.

Proof. It is enough to show every $x \in f^{-1}(U)$ has open quasi-compact neighbourhood V with $f(V) = U$.

Choose affine neighbourhood W' of x with $W' \subseteq f^{-1}(U)$. Choose affine cover $\bigcup W_i = f^{-1}(U)$. We see f is open means $f(W_i)$ are open. Since f is surjective, we see $U = f(f^{-1}(U))$. Thus $U = \bigcup f(W_i)$. However, since U is quasi-compact, so we can assume the index set is finite. Let $V := W' \cup \bigcup_{i \in I} W_i$, we see this is a finite union of affine, so in particular it is quasi-compact.

Definition 1.6.5

We say a property P of morphisms of schemes is ***local on the base (target) for the fppf (et, Zar, etc.) topology***, if for all Cartesian diagrams

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
S' \arrow[r, two heads, "\text{fppf}"] & S
\end{tikzcd}$$

if f' has P then f has P.

Theorem 1.6.6

The following properties are local on the base for the fppf topology: surjective, locally of finite type, locally of finite presentation, of finite type, of finite presentation, universally closed, universally open, separated, proper, unramified, smooth, étale, flat, affine, isomorphism, open immersion, closed immersion, finite, locally quasi-

> *finite, quasi-finite, quasi-compact, quasi-separated, universally injective,universally homeomorphism, and the list goes on.*

Proof. The whole proof is on Stacks Project, Tag 02YJ.

We will only prove some of these.

Universally Closed: we need to prove that in the following diagram

$$\begin{tikzcd} X_T \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r, "\forall"] & S \end{tikzcd}$$

that $X_T \to T$ is closed for all base change. To do this, we consider the following cube of Cartesian:

$$\begin{tikzcd} X'_{T'} \arrow[rrr] \arrow[dr] \arrow[dd, "g'"] & & & X_T \arrow[dr] \arrow[dd] & \\ & X' \arrow[rrr, "g"] \arrow[dd] & & & X \arrow[dd, "f"] \\ T' \arrow[rrr, "f'"] \arrow[dr] & & & T \arrow[dr] & \\ & S' \arrow[rrr] & & & S \end{tikzcd}$$

In the above, all squares are Cartesian. Thus f' is universally closed implies g' is closed. Hence, we get

$$\begin{tikzcd} X'_{T'} \arrow[r, two heads, "\pi'"] \arrow[d, "g'"] & X_T \arrow[d, "g"] \\ T' \arrow[r, two heads, "\pi \quad \text{fppf}"] & T \end{tikzcd}$$

and we want to show g' is closed. In particular, π is fppf implies π is open.

Thus, π surjective and open, we see if $W \subseteq T$, then W is closed iff $\pi^{-1}(W)$ is closed. Thus take $Z \subseteq X_T$ closed, we want $g(Z)$ to be closed. It is enough to show $\pi^{-1}(g(Z))$ is closed. Take

$$Z' := (\pi')^{-1}(Z) = \{(t', z) : z \in Z, \pi(t') = g(z)\}$$

We see Z' is closed and

$$g'(Z') = \{t' : \exists z \in Z \text{ with } \pi(t') = g(z)\}$$

but we see this is just

$$g'(Z') = \pi^{-1}(g(Z))$$

Thus we see $g'(Z')$ is closed because g' is a closed map. This concludes our proof.

Separated: Recall $f : X \to S$ is separated means $\Delta_{X/S} : X \to X \times_S X$ is closed immersion. But Δ is always immersion, thus we just need to show $\Delta_{X/S}$ is closed.

Consider Cartesian diagram

$$\begin{tikzcd}
X_T \arrow[r] \arrow[d] & X \arrow[d] \\
T \arrow[r] & S
\end{tikzcd}$$

We get another diagram

$$\begin{tikzcd}
X_T \arrow[r] \arrow[d, "\Delta_{X_T/T}"] & X \arrow[d, "\Delta_{X/S}"] \\
X_T \times_T X_T \arrow[r] \arrow[d] & X \times_S X \arrow[d] \\
T \arrow[r] & S
\end{tikzcd}$$

Since separated maps base change to be separated, saying $\Delta_{X/S}$ is closed is the same as saying $\Delta_{X/S}$ being universally closed.

Thus, we get

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "\Delta_{X'/S'}"] & X \arrow[d, "\Delta_{X/S}"] \\
X' \times_{S'} X' \arrow[r] \arrow[d] & X_S \times X \arrow[d] \\
S' \arrow[r, "\text{fppf}"] & S
\end{tikzcd}$$

but the bottom arrow is fppf, and hence we see $\Delta_{X'/S'}$ is universally closed implies $\Delta_{X/S}$ is universally closed.

Locally of Finite type: we can reduce to the affine case

$$\begin{tikzcd}
\operatorname{Spec} B' \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
\operatorname{Spec} A' \arrow[r, "\text{fppf}"] & \operatorname{Spec} A
\end{tikzcd}$$

where we have B' is f.g. A-algebra (and the diagram is Cartesian). Let $y_1', \dots, y_m' \in B'$ be generators. We see $B' = A' \otimes_A B$ and hence $y_i' = \sum_j a_{ij}' \otimes x_{ij}$. Let $C \subseteq B$ be the A-algebra generated by the x_{ij}.

By flatness of $A \to A'$, we see

$$C \otimes_A A' \hookrightarrow B \otimes_A A' = B'$$

Thus we see $C \otimes_A A' = B'$. By faithful flatness, we see this forces $C = B$.

Quasi-compact: easy.

Proper: We have

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r, "\text{fppf}"] & S
\end{tikzcd}$$

with f' proper. However, recall proper is the same as universally closed, separated, locally of finite type and quasi-compact. Thus we are done.

1.7 Fppf Topology

Next, we will show if X is a scheme, then h_X is sheaf for the fppf topology.

We expand the definitions to see what this means. We see we need to work with fppf cover $\{Z_i \to Z\}_{i \in I}$, where $Z_i \to Z$ flat locally of finite presentation and $\coprod Z_i \to Z$ surjective. Then we need to show $h_X : (\mathbf{Sch}/\operatorname{Spec}\mathbb{Z})^{\mathrm{opp}}_{\mathrm{fppf}} \to (\mathbf{Sets})$ is a functor, which says it is presheaf. We also need to show exact sequence

$$h_X(Z) \longrightarrow \prod_i h_X(Z_i) \rightrightarrows \prod_{i,j} h_X(Z_i \times_Z Z_j)$$

Then, we claim h_X is a sheaf for fppf topology (i.e. we get the above exact sequence) is the same as to show

$$\begin{tikzcd}
Z_{ij} = Z_i \times_Z Z_j \arrow[r] \arrow[d] & Z_i \arrow[d] \arrow[ddr, bend left] & \\
Z_j \arrow[r] \arrow[drr, bend right] & Z \arrow[dr, dashed, "\exists!"] & \\
& & X
\end{tikzcd}$$

we have unique dashed arrow from Z to X.

Remark 1.7.1

To see those two are the same, we expand what it means to have

$$h_X(Z) = \mathrm{Eq}\left(\prod_i h_X(Z_i) \rightrightarrows \prod_{i,j} h_X(Z_i \times_Z Z_j) \right)$$

Note $h_X(Z) = \mathrm{Hom}(Z, X)$, $h_X(Z_i) = \mathrm{Hom}(Z_i, X)$, and $h_X(Z_{ij}) = \mathrm{Hom}(Z_{ij}, X)$. Since products commutes with what we are doing, we just need to check on a pair of indices i, j. Thus, we get $Z_i \to Z$ and $Z_j \to Z$ by assumption (from the covering), and $Z_{ij} \to Z_i$ and $Z_{ij} \to Z_j$. Then the two arrows $\prod_i h_X(Z_i) \rightrightarrows \prod_{ij} h_X(Z_{ij})$ correspond to the composition $Z_{ij} \to Z_i \to X$ and $Z_{ij} \to Z_j \to X$. Now we want to say that every such pair of arrows come from a unique arrow in $h_X(Z)$, i.e. we want unique $Z \to X$ so the above diagram commutes, as desired.

A special case of this is that, $Z = \bigcup Z_i$ open cover and $f : Z \to X$ is equivalent to $f_i : Z_i \to X$ such that $f_i|_{Z_{ij}} = f_j|_{Z_{ij}}$.

Let's recap.

The big theorem we are working towards at this point is that, if X is a scheme, then h_X is a sheaf for the fppf site on all schemes, i.e. the fppf site on $\operatorname{Spec}\mathbb{Z}$. In particular, if $X \to Y$ then h_X is a sheaf on the fppf site on $(\mathbf{Sch/Y})$.

In the above, we talked about what it means for h_X be a sheaf for fppf topology.

Example 1.7.2

Let L/K be a Galois field extension with Galois group G. Then $\operatorname{Spec} L \to \operatorname{Spec} K$ is étale and hence fppf. We showed for any sheaf $\mathscr{F}$, we see $\mathscr{F}(K) = \mathscr{F}(L)^L$. In particular, taking $\mathscr{F} = h_X$, we see a morphism $\operatorname{Spec} K \to X$ is the same as a G-invariant morphism $\operatorname{Spec} L \to X$.

Since Zariski and étale covers are examples of fppf covers, the big theorem also says h_X is a sheaf for the (big) étale and Zariski topologies. We will start prove the theorem.

Proposition 1.7.3

If $A \to B$ is faithful flat and M is A-module, then

$$M \xrightarrow{f} M \otimes_A B \overset{p_1}{\underset{p_2}{\rightrightarrows}} M \otimes_A B \otimes_A B$$

is exact. Here the maps are given by $f : m \mapsto m \otimes 1$ and $p_1 : m \otimes b \mapsto m \otimes b \otimes 1$ and $p_2 : m \otimes b \mapsto m \otimes 1 \otimes b$.

Proof. Exactness is equivalent to exactness of $0 \to M \xrightarrow{f} M \otimes B \xrightarrow{p_1 - p_2} M \otimes B \otimes B$. Thus it is enough to show (since $A \to B$ is faithfully flat, the old sequence is exact iff we tensor with B) that

$$M \otimes_A B \xrightarrow{f' := f \otimes \mathrm{Id}} M \otimes B \otimes B \overset{p_1'}{\underset{p_2'}{\rightrightarrows}} M \otimes B \otimes B \otimes B$$

where

$$p_1'(m \otimes b \otimes b') = m \otimes b \otimes 1 \otimes b'$$
$$p_2'(m \otimes b \otimes b') = m \otimes 1 \otimes b \otimes b'$$

and $f'(m \otimes b) = f(m) \otimes \mathrm{Id}(b) = m \otimes 1 \otimes b$. The point of doing this is that now we get a section, i.e. we have

$$M \otimes B \xleftarrow[\gamma]{} M \otimes B \otimes B \xleftarrow[\tau]{} M \otimes B \otimes B \otimes B$$

where

$$\tau(m \otimes b \otimes b' \otimes b'') = m \otimes b \otimes b'b''$$
$$\gamma(m \otimes b \otimes b') = m \otimes bb'$$

In particular, we get $\tau p_1' = \mathrm{Id}_{M \otimes B \otimes B}$ and $\tau p_2' = f'\gamma$. Indeed,

$$\tau p_1'(m \otimes b \otimes b') = \tau(m \otimes b \otimes 1 \otimes b') = m \otimes b \otimes 1b' = m \otimes b \otimes b'$$

$$\tau p_2'(m \otimes b \otimes b') = \tau(m \otimes 1 \otimes b \otimes b') = m \otimes 1 \otimes bb'$$

where we note

$$f'\gamma(m \otimes b \otimes b') = f'(m \otimes bb') = m \otimes 1 \otimes bb'$$

One can also check $\gamma f' = \mathrm{Id}$. Hence, we indeed get a section which implies f' is injective as desired (which proves exactness on the left).

Next, we check exactness on the middle. Suppose $\alpha \in M \otimes B \otimes B$, then we have $p_1'(\alpha) = p_2'(\alpha)$. In particular, we get

$$\tau p_1'(\alpha) = \tau p_2'(\alpha) = f'(\gamma(\alpha)) = \alpha$$

This concludes the exactness on the middle as well.

Corollary 1.7.3.1

If U, V, X are affine schemes. If $V \twoheadrightarrow U$ is fppf cover, then we get exact sequence

$$h_X(U) \longrightarrow h_X(V) \rightrightarrows h_X(V \times_U V)$$

Proof. Say $U = \operatorname{Spec} A, V = \operatorname{Spec} B$ and $X = \operatorname{Spec} R$. Then take $M = A$ in previous proposition, we get

$$A \xrightarrow{\iota} B \rightrightarrows B \otimes_A B$$

is exact. In particular, ι is injective. We want that, when we take $\mathrm{Hom}(R, -)$ to the above sequence, we get exact sequence, i.e. we want to show the following sequence is exact

$$\mathrm{Hom}(R, A) \longrightarrow \mathrm{Hom}(R, B) \rightrightarrows \mathrm{Hom}(R, B \otimes_A B)$$

Exact on the left: Suppose we have $R \overset{\alpha}{\underset{\beta}{\rightrightarrows}} A \xrightarrow{\iota} B$ with $\iota\alpha = \iota\beta$ and ι injective. However, this implies $\alpha = \beta$ as ι is injective, as desired.

Exact on the middle: Say $f : R \to B$ such that for all $v \in R$, $f(r) \otimes 1 = 1 \otimes f(r)$. By previous proposition, we know

$$f(r) \in A \subseteq B$$

Hence f factors through A, which proves our claim.

Lemma 1.7.4

Let $\mathscr{F} : (\textbf{Sch})^{opp} \to (\textbf{Sets})$ be a big Zariski sheaf. Then $\mathscr{F}$ is a sheaf for fppf topology *iff for all $V \twoheadrightarrow U$ fppf, we get exact sequence*

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(V) \rightrightarrows \mathscr{F}(V \times_U V)$$

Proof. Clearly if $\mathscr{F}$ is sheaf for fppf topology, then we get the desired exact sequence. We just need to show the converse.

Let $\{U_i \to U\} \in \mathrm{Cov}(U)$ be fppf cover. Let $V = \coprod U_i$. Then we get a sequence

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \times_U U_j)$$

However, we can complete the above diagram to

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] & \prod_i \mathscr{F}(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{ij} \mathscr{F}(U_i \times_U U_j) \\
\mathscr{F}(U) \arrow[r] \arrow[u, "="] & \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[u, "\sim"] & \mathscr{F}(V \times_U V) \arrow[u, "\sim"]
\end{tikzcd}$$

The vertical maps are isomorphisms because U_i form open (Zariski) cover of V, and $U_i \cap U_j = \emptyset$ in V. Thus, if we assume the top row is exact, then we can indeed conclude the bottom row is exact, which implies $\mathscr{F}$ is sheaf for fppf topology, as desired.

Lemma 1.7.5

Let $\mathscr{F} : (\textbf{Sch})^{opp} \to (\textbf{Sets})$ be a presheaf. Assume $\mathscr{F}$ is a sheaf for the big Zariski topology. Then $\mathscr{F}$ is fppf sheaf iff for all $V \twoheadrightarrow U$ fppf, V, U affine, we have

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(V) \rightrightarrows \mathscr{F}(V \times_U V)$$

is exact.

Proof. By previous lemma, it is enough to check sheaf axioms on $V \twoheadrightarrow U$ not necessarily affine but singleton covers.

Exactness on the left: We need to show $\mathscr{F}(U)$ injects into $\mathscr{F}(V)$. Let $U = \bigcup U_i$ be open affine cover and $f : V \twoheadrightarrow U$ be fppf. Then $f^{-1}(U_i) = \bigcup_j V_{ij}$ be an open affine cover. Hence $V = \bigcup_{ij} V_{ij}$. We get the following diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d, hook] & \mathscr{F}(V) \arrow[d, hook] \\
\prod_i \mathscr{F}(U_i) \arrow[r] & \prod_{i,j} \mathscr{F}(V_{ij})
\end{tikzcd}$$

Note the vertical arrows are injective since $\mathscr{F}$ is a Zariski sheaf. Thus, to prove the exactness on the left, it is enough to show the bottom arrow is injective. Since f is flat locally of finite presentation, we see f is open. Thus $f(V_{ij})$ are open in U_i. Also, f is surjective as its fppf, we see

$$U_i = f(f^{-1}(U_i)) = \bigcup_j f(V_{ij})$$

where the each $f(V_{ij})$ are open. Since U_i is affine, it is quasi-compact, we see we can take finite subcover $U_i = \bigcup_{k=1}^{l} f(V_{ij_k})$. In particular, we see the map

$$\coprod_{k=1}^{l} V_{ij_k} \twoheadrightarrow U$$

is fppf because

$$\begin{tikzcd}
V_{ij} \subseteq f^{-1}(U_i) \arrow[r, "\text{fppf}"] \arrow[d, "\subseteq"] & U_i \arrow[d, "\subseteq"] \\
V \arrow[r, "\text{fppf}"] & U
\end{tikzcd}$$

where the inclusion $V_{ij} \subseteq f^{-1}(U_i)$ is flat. Thus we see since $\coprod_{k=1}^{l} V_{ij_k}$ and U_i are both affine, so by assumption, we get

$$\mathscr{F}(U_i) \hookrightarrow \prod_{k=1}^{l} \mathscr{F}(V_{ij_k}) \xrightarrow{\subseteq} \prod_j \mathscr{F}(V_{ij})$$

which shows exactness on the left.

Exactness on the middle: Suppose $V \twoheadrightarrow U$ is fppf with V, U not necessarily affine. We will do step by step.

Step 1: we show we may assume U affine. Let $U = \bigcup U_i$ be affine open cover. Let $V_i = f^{-1}(U_i)$. Then we get

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d, hook, "a"] & \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook, "b"] & \mathscr{F}(V \times_U V) \arrow[d] \\
\prod_i \mathscr{F}(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \mathscr{F}(V_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \mathscr{F}(V_i \times_{U_i} V_i) \\
\prod_{i,j} \mathscr{F}(U_i \cap U_j) \arrow[r, hook, "c"] & \prod_{i,j} \mathscr{F}(V_i \cap V_j) &
\end{tikzcd}$$

We get a, b injective since $\mathscr{F}$ is big Zariski sheaf, the c is injective since $V_i \cap V_j \twoheadrightarrow U_i \cap U_j$ is fppf and we apply exactness on the left shown above. A diagram chase shows exactness in the middle as desired (if we can show the U affine case), i.e. we want $\mathscr{F}(V) \rightrightarrows \mathscr{F}(V \times_U V)$ to be exact.

The diagram chase is roughly as follows: start with the top middle bullet $\bullet_1$, we want to ask if there exists $\bullet_?$ in $\mathscr{F}(U)$ that maps to $\bullet_1$:

$$\bullet_? \in \mathscr{F}(U) \xrightarrow{?} \bullet_1 \in \mathscr{F}(V) \rightrightarrows \bullet_2 = \bullet_3 \in \mathscr{F}(V \times_U V)$$

There is not much we can do at this point, thus we send $\bullet_1$ to the bottom via b and get

$$\begin{tikzcd}
\bullet_? \in \mathscr{F}(U) \arrow[r, "?"] & \bullet_1 \in \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \bullet_2 = \bullet_3 \in \mathscr{F}(V \times_U V) \\
 & \bullet_b &
\end{tikzcd}$$

We want to show $\bullet_b$ comes from $\prod \mathscr{F}(U_i)$. Thus we get

$$\begin{tikzcd}
\bullet_? \in \mathscr{F}(U) \arrow[r, "?"] \arrow[d, hook, "?"] & \bullet_1 \in \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \bullet_2 = \bullet_3 \in \mathscr{F}(V \times_U V) \arrow[d] \\
\bullet_a \arrow[r, "?"] & \bullet_b \arrow[r, shift left] \arrow[r, shift right] & \bullet_{2b} = \bullet_{3b}
\end{tikzcd}$$

However, note the middle row is exact by assumption, we indeed get

$$\begin{tikzcd}
\bullet_? \in \mathscr{F}(U) \arrow[r, "?"] \arrow[d, hook, "?"] & \bullet_1 \in \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \bullet_2 = \bullet_3 \in \mathscr{F}(V \times_U V) \arrow[d] \\
\bullet_a \arrow[r] & \bullet_b \arrow[r, shift left] \arrow[r, shift right] & \bullet_{2b} = \bullet_{3b}
\end{tikzcd}$$

Viz, we have $\bullet_b$ comes from $\bullet_a$ and it remains to show $\bullet_a$ comes from the injection ? arrow from $\mathscr{F}(U)$. To that end, we note the left vertical line is exact, hence to show $\bullet_a$ lives in the image of $\mathscr{F}(U)$, we just need to show $\bullet_a$ has the same image in $\prod_{i,j} \mathscr{F}(U_i \cap U_j)$. To show that, we map $\bullet_a$ forward via the two different maps, and get $\bullet_{a2}$ and $\bullet_{a3}$, i.e. we get

$$\begin{tikzcd}
\bullet_? \in \mathscr{F}(U) \arrow[r, "?"] \arrow[d, hook, "?"] & \bullet_1 \in \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \bullet_2 = \bullet_3 \in \mathscr{F}(V \times_U V) \arrow[d] \\
\bullet_a \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \bullet_b \arrow[r, shift left] \arrow[r, shift right] & \bullet_{2b} = \bullet_{3b} \\
\bullet_{a2}, \bullet_{a3} \arrow[r, hook] & {} &
\end{tikzcd}$$

where at the bottom, we must have $\bullet_{a2}$ and $\bullet_{a3}$ map to the same element because the middle column $\mathscr{F}(V) \to \prod_i \mathscr{F}(V_i) \rightrightarrows \prod_{i,j} \mathscr{F}(V_i \cap V_j)$ is exact and the image of $\bullet_{a2}$ and $\bullet_{a3}$ must equal the image of $\bullet_1$. Hence, this forces $\bullet_{a2} = \bullet_{a3}$ which forces $\bullet_a$ to come from $\bullet_?$ and hence shows $\bullet_1$ indeed comes from $\bullet_?$ as desired.

After this point, we assume U is affine.

Step 2: we show we can assume V is quasi-compact. We showed last time there exists $V = \bigcup V_j$ open cover by quasi-compacts such that $V_j \twoheadrightarrow U$ fppf. Consider the restriction map (for each j)

$$\begin{tikzcd}
x \in \arrow[d, mapsto] & \mathrm{Eq}(\mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathscr{F}(V \times_U V)) \arrow[d] \\
x_j \in & \mathrm{Eq}(\mathscr{F}(V_j) \arrow[r, shift left] \arrow[r, shift right] & \mathscr{F}(V_j \times_U V_j))
\end{tikzcd}$$

Our goal is to show x comes from $\mathscr{F}(U)$, where we assume that this x comes from $\mathscr{F}(U)$ when V is quasi-compact.

Hence, assume quasi-compact case. Since $V_j \twoheadrightarrow U$ is fppf, we get the following sequence

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(V_j) \rightrightarrows \mathscr{F}(V_j \times_U V_j)$$

is exact. Thus there exists unique $y_j \in \mathscr{F}(U)$ mapping to x_j. We claim y_j is independent of j. Indeed, consider the diagram

$$\begin{tikzcd}
V_i \times_U V_j \arrow[r, two heads] \arrow[d, two heads] & V_j \arrow[d, "\subseteq"] \arrow[dr, two heads, bend left] & \\
V_i \arrow[r, "\subseteq"] \arrow[drr, two heads, bend right] & V \arrow[dr, two heads] & \\
& & U
\end{tikzcd}$$

where we used fppf maps $V_j \to U$ and $V_i \to U$ to get the fibered product. Now we apply $\mathscr{F}$ to the whole diagram. First, we get the following injections

$$\begin{tikzcd}
\mathscr{F}(V_i \times_U V_j) & \mathscr{F}(V_j) \arrow[l, hook'] & \\
\mathscr{F}(V_i) \arrow[u, hook] & \mathscr{F}(V) \arrow[l, hook'] \arrow[u, hook] & \\
& & \mathscr{F}(U) \arrow[ul, hook'] \arrow[ull, hook'] \arrow[uul, hook']
\end{tikzcd}$$

Now let's chase elements:

$$\begin{tikzcd}
\mathscr{F}(V_i \times_U V_j) & x_j \in \mathscr{F}(V_j) \arrow[l, hook'] & \\
x_i \in \mathscr{F}(V_i) \arrow[u, hook] & x \in \mathscr{F}(V) \arrow[l, hook'] \arrow[u, hook] & \\
& & \mathscr{F}(U) \arrow[ul, hook'] \arrow[uul, hook', bend right, "y_j \mapsto x_j"'] \arrow[ull, bend left, "y_i \mapsto x_i"]
\end{tikzcd}$$

However, since x maps to x_i and x_j, we know x_i and x_j must map to the same thing in $\mathscr{F}(V_i \times_U V_j)$. However, $V_i \times_U V_j \to V_i \to U$ is fppf cover, thus the two arrows $\mathscr{F}(U) \rightrightarrows \mathscr{F}(V_i \times_U V_j)$ are injective. Thus, we must have $y_i = y_j$, hence we can denote this as $y = y_i = y_j$. Moreover we have $y \mapsto x$. This is exactly what we wanted, and hence this finishes step 2.

After this step, we assume U is affine awnd V is quasi-compact.

Step 3: finish the proof. We may assume $V \twoheadrightarrow U$ fppf with V quasi-compact and U affine. Let $V = \bigcup V_j$ be a finite affine cover. In particular, since the union is finite, we see $\coprod V_j$ is affine and hence $\coprod V_j \twoheadrightarrow U$ is fppf.

Thus we get

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d, "="] & \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \mathscr{F}(V \times_U V) \arrow[d, hook] \\
\mathscr{F}(U) \arrow[r] & \mathscr{F}(\coprod_j V_j) \arrow[r, shift left] \arrow[r, shift right] & \mathscr{F}(\coprod_j V_j \times_U \coprod_j V_j)
\end{tikzcd}$$

The vertical arrows are injective since $\coprod V_j \twoheadrightarrow V$ is fppf cover. The bottom row is exact because $\coprod_j V_j \twoheadrightarrow U$ is fppf and both of them are affine and we are assuming the affine case holds. Hence the top row is exact. This concludes the proof.

Corollary 1.7.5.1

If X is affine, then h_X is fppf sheaf.

Proof. We proved sheaf axiom for $V \twoheadrightarrow U$ where V, U are affine and the arrow is fppf. Also, it is easy to check h_X is big Zariski sheaf. Hence h_X is fppf sheaf.

At this point, we proved a big lemma says if $\mathscr{F}$ is sheaf for big Zariski topology, then $\mathscr{F}$ is sheaf for fppf topology iff for affine singleton covers we get the exact sequence.

The next step is to show we can move from affine X to any X.

Theorem 1.7.6

Let X be a scheme, then h_X is a fppf sheaf.

Proof. Let $X = \bigcup_i X_i$ be open affine cover. Let $V \twoheadrightarrow U$ be fppf with U, V affine. We just need to show

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(V) \rightrightarrows \mathscr{F}(V \times_U V)$$

is exact (where of course $\mathscr{F} = h_X$).

Exact on the left: pick $f, g \in h_X(U)$ with f, g maps to the same element in $\mathscr{F}(V)$. That is, say we have

$$\begin{tikzcd} V \arrow[r, two heads, "t"] & U \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \end{tikzcd}$$

such that $f t = g t$. We want $f = g$. Set-theoretically, we know $f = g$ as t is surjection. Thus we just need to show this equality is scheme-theoretically. Since $X = \bigcup X_i$ and $f = g$ as set maps, we see $f^{-1}(X_i) = g^{-1}(X_i)$ as sets. Thus let's define $U_i := f^{-1}(X_i) = g^{-1}(X_i)$. Now take the diagram and restrict to U_i with $V_i := t^{-1}(U_i)$, we get

$$\begin{tikzcd} V_i \arrow[r, two heads, "t|_{V_i}", "\text{fppf}"'] & U_i \arrow[r, shift left, "f|_{U_i}"] \arrow[r, shift right, "g|_{U_i}"'] & X_i \end{tikzcd}$$

In particular, we get $f|_{U_i} \circ t|_{V_i} = g|_{U_i} \circ t|_{V_i}$. However, h_{X_i} is a sheaf as X_i is affine, thus we see $f|_{U_i} = g|_{U_i}$ scheme-theoretically. However, this holds for all U_i and hence we see $f = g$ scheme-theoretically globally.

Exact on the middle: say we have

$$\begin{tikzcd} V \times_U V \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & V \arrow[r, two heads, "t"] \arrow[dr, "f"] & U \arrow[d, dotted, "\exists h"] \\ & & X \end{tikzcd}$$

with $f p_1 = f p_2$. We want to show the dash arrow exists, i.e. we want to show there exists h. Let $|T|$ be the underlying topological space of any scheme T. Then we see we get the same diagram for topological spaces

$$\begin{tikzcd} {|V \times_U V|} \arrow[r, shift left, "|p_1|"] \arrow[r, shift right, "|p_2|"'] & {|V|} \arrow[r, "|t|"] \arrow[dr, "|f|"] & {|U|} \arrow[d, dotted, "\exists h"] \\ & & {|X|} \end{tikzcd}$$

However, in this case, $|h|$ exists because of the following claim.

Claim:

$$\begin{tikzcd} {|V \times_U V|} \arrow[r, shift left, "|p_1|"] \arrow[r, shift right, "|p_2|"'] & {|V|} \arrow[r, "|t|"] & {|U|} \end{tikzcd}$$

is a coequalizer of topological spaces.

Suppose this claim holds, then h exists topologically , and so we can talk about subschemes $U_i := h^{-1}(X_i) \subseteq U$, $V_i := f^{-1}(X_i) \subseteq V$. Then, we get

$$\begin{tikzcd} V_i \arrow[r, "\text{fppf}", "t|_{V_i}"'] \arrow[dr, "f|_{V_i}"'] & U_i \arrow[d, dotted, "\exists! h_i"] \\ & X_i \end{tikzcd}$$

where the existence of h_i is by affine case. Moreover, we see $h_i|_{U_i \cap U_j} = h_j|_{U_i \cap U_j}$ because we can cover $X_i \cap X_j$ by affine open and h_i restrict to open affine agrees with h_j restricts to open affine by the uniqueness statement in the affine case. Since $h_i|_{U_i \cap U_j} = h_j|_{U_i \cap U_j}$, the h_i's glue to a map $h : U \to X$ scheme-theoretically. Thus, if we can prove the above claim, we are done. We are going to prove it in small steps.

Next, we claim that for fppf $t : V \twoheadrightarrow U$,:

1. there exists natural surjection $|V \times_U \times V| \twoheadrightarrow |V| \times_{|U|} |V|$
2. $R \subseteq |U|$ open iff $|t|^{-1}(R) \subseteq |V|$ is open.
3. $\begin{tikzcd} {|V \times_U V|} \arrow[r, shift left, "|p_1|"] \arrow[r, shift right, "|p_2|"'] & {|V|} \arrow[r, "|t|"] & {|U|} \end{tikzcd}$ is coequalizer in category of topological spaces.

(1): Take $x, x' \in |V|$ with the same image $\overline{x} \in |U|$. Then we see we get

$$\begin{tikzcd} \operatorname{Spec} \kappa(x) \arrow[drr, bend left] \arrow[ddr, bend right, two heads] & & \\ & \operatorname{Spec} \kappa(x') \arrow[r] \arrow[d, two heads] & V \arrow[d, "t"] \\ & \operatorname{Spec} \kappa(\overline{x}) \arrow[r] & U \end{tikzcd}$$

What we do next is to take fibered products of the two residue fields over $\operatorname{Spec}\kappa(\overline{x})$ and we get

$$\operatorname{Spec}\kappa(x) \times_{\operatorname{Spec}(\overline{x})} \operatorname{Spec}\kappa(x') \longrightarrow V \times_U V$$

Here the arrow above is the dashed arrow (by universal property) in the following

$$\begin{tikzcd}
\operatorname{Spec}\kappa(x) \times_{\operatorname{Spec}\kappa(\overline{x})} \operatorname{Spec}\kappa(x') \arrow[r] \arrow[d] \arrow[drr, dashed] & \operatorname{Spec}\kappa(x) \arrow[r] & V \arrow[dd, "t"] \\
\operatorname{Spec}\kappa(x') \arrow[d] & & V \times_U V \arrow[u] \arrow[dll] \\
V \arrow[rr, "t"] & & U
\end{tikzcd}$$

In particular, we see

$$\operatorname{Spec}\kappa(x) \times_{\operatorname{Spec}(\overline{x})} \operatorname{Spec}\kappa(x') = \operatorname{Spec}(\kappa(x) \otimes_{\kappa(\overline{x})} \kappa(x'))$$

and we just choose any point of $\operatorname{Spec}(\kappa(x) \otimes_{\kappa(\overline{x})} \kappa(x'))$ and its image inside $V \times_U V$ would be a point that correspond to (x, x') in $|V| \times_{|U|} |V|$. This yields a point of $|V \times_U V|$ mapping to $(x, x') \in |V| \times_{|U|} |V|$.

(2): $R \subseteq |U|$ is open then since t is continuous we get $|t|^{-1}(R)$ is open. Conversely, t is fppf implies it is surjective, thus $R = t(t^{-1}(R))$ and hence its open as $t^{-1}(R)$ is open and t is open map.

(3): we need to show the diagram is a coequalizer diagram. In this part, we will drop the bars, and just move to the category of topological spaces. What we want is that for any W topological space, we want

$$\begin{tikzcd}
V \times_U V \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & V \arrow[r, "t"] \arrow[dr, "f"] & U \arrow[d, dotted, "\exists! h"] \\
& & W
\end{tikzcd}$$

If h exists, then it is unique as t is surjective, i.e. $u \in |U|$, then choose $v \in |V|$ such that $t(v) = u$, then $h(u) = f(v)$.

Thus we just need to show if $v, v' \in |V|$ and $t(v) = t(v')$, then $f(v) = f(v')$. However, since $v, v' \in |V|$ with $t(v) = t(v')$, this means $(v, v') \in |V| \times_{|U|} |V|$. Earlier, we showed there is surjection

$$q : |V \times_U V| \twoheadrightarrow |V| \times_{|U|} |V|$$

Let $v'' \in |V \times_U V|$ so $q(v'') = (v, v')$. Then, we see we know $f p_1 = f p_2$ by assumption, thus we see $f p_1(v'') = f p_2(v'')$ but by definition $f p_1(v'') = f(v)$ and $f p_2(v'') = f(v')$ (this is universal property of fibered product on cat of top spaces etc and the fact our map $|V \times V| \to |V| \times |V|$ is natural).

At this point, we have defined h as set map, and we need to show h is continuous. Thus take $W' \subseteq W$ be open, then we see $h^{-1}(W') \subseteq U$ is open iff $t^{-1}h^{-1}(W')$ is open in $|V|$ by part (2). However,

$$t^{-1}h^{-1}(W') = f^{-1}(W')$$

where f is continuous, hence h is continuous as desired.

Chapter 2

Fibered Category

归去来兮，吾归何处？万里家在岷峨。百年强半，来日苦无多。坐见黄州再闰，儿童尽楚语吴歌。山中友，鸡豚社酒，相劝老东坡。
云何，当此去，人生底事，来往如梭。待闲看秋风，洛水清波。好在堂前细柳，应念我，莫剪柔柯。仍传语，江南父老，时与晒渔蓑。

苏轼

Before we start the math, let's talk about intuition[1]. In the introduction section of chapter 1, we mentioned that one of the reason to consider stacks is because we want to study moduli problems.

So, let's go through the mental process of defining moduli spaces:

1. We want to study a class of objects modulo some equivalence conditions.
    - For example, consider the class of triangles Δ, modulo the relation $\Delta_1 \sim_1 \Delta_2$ if they are similar triangles. This gives a "moduli set" $(\Delta, \sim_1)$. Another moduli set would be $(\Delta, \sim_2)$ where $\sim_2$ is actual equality.
2. But we also want to add additional structure to our "moduli sets", so they the moduli set themselves have geometry, i.e. we get moduli spaces.
    - For example, the moduli space of plane conics is $\mathbb{P}^5$, and it has very rich geometry
3. However, the million dollar question is that, what additional structure is appropriate? This is because we already see it cannot be a scheme structure, as the moduli space of vector bundles cannot be a scheme.

In our approach of defining moduli spaces using stacks, given moduli set $(S, \sim)$, we enrich S to be a category $\mathcal{S}$, and change $\sim$ to collection of isomorphisms between

[1]The following is from Alper's book stacks and modulis

objects of $\mathcal{S}$. This is precisely the definition of groupoids:

Definition

A category $\mathcal{S}$ is a ***groupoid*** if every morphism is an isomorphism.

We call this a moduli groupoid if it classifies some objects that we care about. We already seen a lot of examples of such moduli groupoids.

Example

Let G be a group acts on set X, then the moduli groupoid of orbits $[X/G]$ is the category with objects being $x \in X$ and $\mathrm{Mor}(x, x') = \{g \in G : x' = gx\}$.

For example, let's consider $[\mathbb{A}^1/(\mathbb{Z}/2)]$ where $g \cdot x = gx$ is scalar multiplication. In this case we see if $x \neq 0$ then $\mathrm{Mor}(x, x) = \{1\}$, and $\mathrm{Mor}(0, 0) = \mathbb{Z}/2\mathbb{Z}$. In other word, $[\mathbb{A}^1/(\mathbb{Z}/2)]$ is basically $\mathbb{A}^1$, except at the origin, where we have non-trivial automorphism group.

Another instance of moduli groupoid of orbits is $\mathbb{P}^n = [(\mathbb{A}^{n+1} \backslash 0)/\mathbb{G}_m]$.

But now let's mix the cocktail even further and add the ingredients we cooked in chapter 1:

1. Recall our endgame for moduli spaces is to define a theory so that our moduli spaces are geometric spaces themselves, i.e. the best case is we get a scheme, like the moduli space of plane conics.
2. But what is a geometric space, well, from chapter one we know it is just a representable functor.
3. Thus, our definition is getting more complicated. A moduli functor is just a functor $F : \mathcal{C} \to (\mathbf{Sets})$ where $\mathcal{C}$ is some category. For now we just stick with $\mathcal{C} = (\mathbf{Sch})$, but clearly we can have $(\mathbf{Sch})_{\text{ét}}$ and all that.
4. But our problem is that the $\mathrm{Mor}(x, x')$ in our case might not be just sets! We want more structures, we need to enrich it (again) to groupoids. That is, we need a functor $F : \mathcal{C} \to (\mathbf{Grpoids})$ where $(\mathbf{Grpoids})$ is the "category" of groupoids.

This leads to the notion of fibered category, which we are going to define next. We remark that the definition we give will first look rather weird, this is because in the actual definition we packed all the images (they are groupoids) $F(C)$ into a massive category, and call it the fibered category.

2.1 Fibered Caregory

From last chapter, we talked about descents. Then, descents plus fibered category gives categorical stacks, then plus geometry then we get algebraic stacks. In particular, categorical stacks are special kind of fibered categories which satisfy descent.

Frequently, given a diagram

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

we say take the fibered product and we get

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

(where we use $\square$ to indicate Cartesian diagram) However, $X \times_Y Z$ is only defined up to (canonical) isomorphism.

Thus, any two objects W_1, W_2 that claim to be the fibered product are isomorphic up to unique isomorphism. Usually it is good enough to make a choice between W_1 and W_2 and the choice does not make a difference.

However, since stacks are about automorphisms, we need to keep track of the choices we made, which is bad. Thus, we can speak of when

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is "a" Cartesian diagram, rather than saying W is "the" fibered product.

Definition 2.1.1

Let $\mathcal{C}$ be a category, then a ***category over*** $\mathcal{C}$ is a category $\mathcal{F}$ and a functor $p : \mathcal{F} \to \mathcal{C}$.

A morphism $\phi : U \to V$ in $\mathcal{F}$ is ***Cartesian*** if for the following diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, mapsto] \arrow[dr, phantom, "\square"] & V \arrow[d, mapsto] \\ p(U) \arrow[r] & p(V) \end{tikzcd}$$

and for all $\psi : W \to V$ and factorization $p(W) \xrightarrow{h} p(U) \xrightarrow{p(\phi)} p(V)$, there exists unique $\lambda : W \to U$ with

$$\begin{tikzcd} W \arrow[r, dotted, "\exists!\lambda"] \arrow[rr, bend left, "\forall\psi"] \arrow[d, mapsto] & U \arrow[r, "\phi"] \arrow[d, mapsto] & V \arrow[d, mapsto] \\ p(W) \arrow[r, "h"] & p(U) \arrow[r] & p(V) \end{tikzcd}$$

such that $\phi\lambda = \psi$ and $p(\lambda) = h$.

The point of this is that, with this, the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, mapsto] & V \arrow[d, mapsto] \\ p(U) \arrow[r] & p(V) \end{tikzcd}$$

looks like a pullback (and it indeed satisfies the universal property of pullbacks), i.e. we are mimicing the definition of pullback here.

Definition 2.1.2

Let $\mathcal{F}$ be category over $\mathcal{C}$ and $\phi : U \to V$ be Cartesian, then we say U is a ***pullback of*** V ***along*** $p(\phi)$.

If $U' \xrightarrow{\phi'} V$ and $U \xrightarrow{\phi} V$ are both pullbacks along the same thing $p(\phi) = p(\phi')$, then we have

$$\begin{tikzcd} U' \arrow[r, dotted, "\exists!\lambda"] \arrow[rr, bend left, "\phi'"] \arrow[d, mapsto] & U \arrow[r, "\phi"] \arrow[d, mapsto] & V \arrow[d, mapsto] \\ p(U') \arrow[r, "\mathrm{Id}"] & p(U) \arrow[r] & p(V) \end{tikzcd}$$

and hence $p(\lambda) = \mathrm{Id}$ which implies λ is an isomorphism (with $\phi' \circ \lambda = \phi$).

Remark 2.1.3

Now given $p : \mathcal{F} \to \mathcal{C}$, for any $U \in \mathcal{C}$, let $\mathcal{F}(U)$ be a category defined as follows: the objects are $X \in \mathcal{F}$ such that $p(X) = U$ and morphisms are $X \xrightarrow{\phi} Y$ such that $p(\phi) = \mathrm{Id}_U$.

The idea above is that we are supposed to think of $\mathcal{F}$ as a map from $\mathcal{C}$ to categories, where we input U and output a "category" $\mathcal{F}(U)$. In other word, this is the notion we want in the introduction, i.e. we get a functor $\mathcal{F} : \mathcal{C} \to$ (**Grpoids**).

Definition 2.1.4

Let $p : \mathcal{F} \to \mathcal{C}$ be a category over $\mathcal{C}$, then we say $\mathcal{F}$ is a ***fibered category*** if "pullbacks exist", i.e. given diagram

$$\begin{tikzcd} & v \arrow[d, mapsto] \\ U \arrow[r, "h"] & V := p(v) \end{tikzcd}$$

then there exists Cartesian arrow $\phi : u \to v$ such that $p(\phi) = h$, i.e.

$$\begin{tikzcd} u \arrow[r, "\phi"] \arrow[d] \arrow[dr, phantom, "\square"] & v \arrow[d] \\ U \arrow[r, "h"] & V \end{tikzcd}$$

is a pullback.

Example 2.1.5

Let $\mathcal{C} = (\mathbf{Sch})$ be the category of schemes, and let M_g be the category of genus g curves. In other words, objects of M_g are $C \xrightarrow{\pi} S$ where π is smooth and geometric fibers of π are genus g curves(recall geometric fiber means $X \times \operatorname{Spec}(\overline{\kappa(x)})$). The morphisms are diagrams

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "\pi'"] & C \arrow[d, "\pi"] \\ S' \arrow[r] & S \end{tikzcd}$$

Then, our $p : M_g \to (\mathbf{Sch})$ is going to be $(C \xrightarrow{\pi} S) \mapsto S$. Since pullback exists because they exists in $(\mathbf{Sch})$, we see M_g is a fibered category. Indeed, just take the fibered product in $(\mathbf{Sch})$, say

$$\begin{tikzcd} & C \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

then we get

$$\begin{tikzcd} C \times_S S' =: C' \arrow[r] \arrow[d, "\pi'"] & C \arrow[d, "\pi"] \\ S' \arrow[r] & S \end{tikzcd}$$

where since π is smooth, then π' is smooth, and π' has the same geometric fibers as π.

Thus, we just constructed the moduli space of genus g curves.

Example 2.1.6

If $\mathcal{C}$ is any category, $X \in \mathcal{C}$, then consider $\mathcal{C}/X$ as the category with objects $Y \to X$ and morphisms

$$\begin{tikzcd} Y' \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd}$$

Then $p : \mathcal{C}/X \to \mathcal{C}$ with $Y \mapsto Y$ gives $\mathcal{C}/X$ the structure of a fibered category. This is because every arrow in $\mathcal{C}/X$ is Cartesian. Indeed, take an arrow $\phi : Y' \to Y$ in $\mathcal{C}/X$, and let $\psi : Y'' \to Y$ be any arrow in $\mathcal{C}/X$ with a factorization $Y'' \xrightarrow{h} Y' \xrightarrow{\phi} Y$, then we see we indeed have dashed arrow in the following diagram

$$\begin{tikzcd}
Y'' \arrow[r, dashed] \arrow[rr, bend left, "\psi"] \arrow[d, "\mathrm{Id}"] & Y' \arrow[r, "\phi"] \arrow[d, "\mathrm{Id}"] & Y \arrow[d, "\mathrm{Id}"] \\
Y'' \arrow[r, "h"] & Y' \arrow[r, "\phi"] & Y
\end{tikzcd}$$

That is, we just take the dashed arrow be h, which is indeed unique and it exists.

Example 2.1.7

Let's consider the category (**QCoh**) over (**Sch**). Here objects of (**QCoh**) are pairs $(S, \mathscr{F})$ where $\mathscr{F}$ is quasi-coherent sheaf on S, and morphisms between $(S', \mathscr{F}') \to (S, \mathscr{F})$ are given by $f : S' \to S$ and $\epsilon : \mathscr{F}' \to f^*\mathscr{F}$.

Example 2.1.8

Let G be a group, considered as a category with one objects, and set of arrows is G itself. A group homomorphism $G \to H$ then can be considered as a functor. Then, an arrow in G is always Cartesian, and hence G is fibered over H if and only if $G \to H$ is surjective.

Example 2.1.9

Consider the forgetful functor $p : (\mathbf{Top}) \to (\mathbf{Set})$ that associates topological space X to the underlying set $p(X)$. This makes (**Top**) a fibered category over (**Set**). Suppose we have a topological space Y, a set U and a function $f : U \to p(Y)$. Denote by X the set U with the initial topology, i.e. open sets are inverse images of the open subsets of Y. If T is a topological space, a function $T \to X$ is continuous iff $T \to X \to Y$ is continuous. This shows $f : X \to Y$ is Cartesian arrow over the given arrow $f : U \to p(Y)$.

The fiber of (**Top**) over a set U is the partially ordered set of possible topologies on U, make into a category in the usual way.

In the above examples, as in Remark 2.1.3, we see $(\mathbf{QCoh})(S)$ is exactly the category of quasi-coherent sheaves on S, and $M_g(S)$ is exactly the category of genus g curves on S.

Indeed, $(\mathbf{Qcoh})(S)$ is by definition the category with objects being $(X, \mathscr{F}) \in (\mathbf{Qcoh})$ such that $p(X, \mathscr{F}) = S$ and morphisms being $f : (X, \mathscr{F}) \to (Y, \mathscr{G})$ such that $p(f) = \mathrm{Id}_S$. Well, $p(f, \epsilon) = \mathrm{Id}_S$ means we need to have $f : X \to Y$ is the identity, i.e. $X = Y = S$ and ϵ is just a morphism between quasi-coherent sheaves on S.

Similarly $M_g(S)$ is category of genus g curves because the projection forces any object to be live over S.

Definition 2.1.10

If $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$ and $p_{\mathcal{G}} : \mathcal{G} \to \mathcal{C}$ be two fibered categories, then a ***morphism of fibered categories*** is a functor $g : \mathcal{F} \to \mathcal{G}$ such that

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "g"] \arrow[dr, "p_{\mathcal{F}}"] & \mathcal{G} \arrow[d, "p_{\mathcal{G}}"] \\
& \mathcal{C}
\end{tikzcd}$$

and g sends Cartesian arrows to Cartesian arrows.

We note, for all $U \in \mathcal{C}$, we get $g_U : \mathcal{F}(U) \to \mathcal{G}(U)$. Indeed, since $\mathcal{F}(U), \mathcal{G}(U)$ are categories, g_U is a functor between categories. It is just the same as g, i.e. $g_U(x) = g(x)$. We can check this is indeed a functor. Let $x_1 \to x_2 \in \mathcal{F}(U), y_1 \to y_2 \in \mathcal{G}(U)$ be two arrows, we need to show we get a diagram

$$\begin{tikzcd}
x_1 \arrow[r, "\phi"] \arrow[d, "g_U"] & x_2 \arrow[d, "g_U"] \\
y_1 \arrow[r, "\psi"] & y_2
\end{tikzcd}$$

However, note we get the following diagram

$$\begin{tikzcd}
x_1 \arrow[r, "\phi"] \arrow[d, dashed, "g_U"] \arrow[dd, bend right, "p_{\mathcal{F}}"'] & x_2 \arrow[d, dashed, "g_U"'] \arrow[dd, bend left, "p_{\mathcal{G}}"] \\
y_1 \arrow[r, "\psi"] \arrow[d, dashed, "p_{\mathcal{G}}"] & y_2 \arrow[d, dashed, "p_{\mathcal{G}}"'] \\
U \arrow[r, "\mathrm{Id}"] & U
\end{tikzcd}$$

where the two triples of vertical arrows (i.e. $(g_U, p_{\mathcal{G}}, p_{\mathcal{F}})$) commutes, and the outer square and inner square both commutes, which forces the upper square to commute as desired.

Remark 2.1.11

We defined $\mathcal{F}(U)$ for any functor $p : \mathcal{F} \to \mathcal{G}$, without the assumption $\mathcal{F}$ is fibered over $\mathcal{C}$. However, if we do not assume $\mathcal{F}$ is fibered over $\mathcal{C}$, then this notion of $\mathcal{F}(U)$ is not very useful. For example, it may happen that we have two objects U and V of $\mathcal{C}$ which are isomorphic, but such that $\mathcal{F}(U)$ is empty while $\mathcal{F}(V)$ is not. This will not happen in fibered categories, precisely because in this case we get functor $g_U : \mathcal{F}(U) \to \mathcal{G}(U)$, as defined above.

Definition 2.1.12

If $g, g' : \mathcal{F} \to \mathcal{G}$ are two morphisms of fibered categories, then a ***base-preserving natural transformation*** $\alpha : g \to g'$ is a natural transformation of functors such that for all $U \in \mathcal{F}$, the map $\alpha_U : g(U) \to g'(U)$ satisfies $p_{\mathcal{G}}(\alpha_U) = \mathrm{Id}_{p_{\mathcal{F}}(U)}$, i.e. if we have the following diagram

$$\begin{tikzcd}
U \arrow[r, mapsto] \arrow[rr, bend left] \arrow[dr, "p_{\mathcal{F}}"'] & g(U) \arrow[r, "\alpha_U"] \arrow[d, mapsto, "p_{\mathcal{G}}"] & g'(U) \arrow[d, mapsto, "p_{\mathcal{G}}"] \\
 & p_{\mathcal{F}}(U) \arrow[r, "\mathrm{Id}_{p_{\mathcal{F}}(U)}"'] & p_{\mathcal{F}}(U)
\end{tikzcd}$$

then we must have $p_{\mathcal{G}}(\alpha_U) = \mathrm{Id}$. In other word, we want α_U to be a morphism in $\mathcal{G}(p_{\mathcal{F}}(U))$.

The reason why we draw the extra $p_{\mathcal{F}}(U) \xrightarrow{\mathrm{Id}} p_{\mathcal{F}}(U)$ is because from this we see we simply want an arrow between the two arrows a_U and $\mathrm{Id}_{p_{\mathcal{F}}(U)}$.

We note this gives $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$ the category with objects being morphisms of fibered category $g : \mathcal{F} \to \mathcal{G}$ and morphisms being base-preserving natural transformations.

Now, suppose we have $g' : \mathcal{F} \to \mathcal{G}$, then we get $g_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for all $U \in \mathcal{C}$. This is kind of looks like a map of presheaves.

At this point, we defined a fibered category, which is a functor $p : \mathcal{F} \to \mathcal{C}$ such that pullbacks exist, i.e. for all z and for all $z \to y$, there exists unique $z \to x$ so the following commutes

$$\begin{tikzcd}
z \arrow[r, dotted, "\exists!"] \arrow[rr, bend left, "\forall"] \arrow[d] & x \arrow[r] \arrow[d] & y \arrow[d] \\
p(z) \arrow[r, "\forall"] & p(x) \arrow[r] & p(y)
\end{tikzcd}$$

The morphisms of fibered categories are given by functors such that g sends Cartesian arrows to Cartesian arrows.

Then, for $p : \mathcal{F} \to \mathcal{C}$ fibered category, for all $U \in \mathcal{C}$, we let $\mathcal{F}(U)$ be the category with objects being $x \in \mathcal{F}$ such that $p(x) = U$, and morphisms being $\phi : x \to y$ such that $p(\phi) = \mathrm{Id}_U$.

Thus, say $g : \mathcal{F} \to \mathcal{G}$ be maps of fibered categories over $\mathcal{C}$, then we get $g_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for all $U \in \mathcal{C}$.We mentioned this means fibered categories look like presheaves but instead of $\mathcal{F}(U)$ being sets, we have $\mathcal{F}(U)$ is category. In particular, we can recover the moduli space of sheaves[2] on S just like how we defined the fibered category $(\mathbf{QCoh}) \to (\mathbf{Sch})$. In light of this, the thing we are going to define next, i.e. categorical stacks, are just axiomitizations of the two sheaf axioms, i.e. we want gluing and restriction axioms on our fibered categories (and this in turn is just descent).

Example 2.1.13

Consider $\mathcal{M}_{g,n} \to (\mathbf{Sch})$ be the fibered category of genus g curves with n marked points. In this category, objects of $\mathcal{M}_{g,n}$ are $C \xrightarrow{\pi} S$ with π smooth and on geo-

[2]That is, consider $(\mathbf{Sh}) \to (\mathbf{Sch})$ with objects being $(\mathscr{F}, S)$ and $p(\mathscr{F}, S) = S$, where $\mathscr{F}$ is any sheaf on S. This is a fibered category and $(\mathbf{Sh})(S)$ is indeed the category of sheaves on S

metric fibers of S, C is genus g curves. Next, we need to explain what marked points are. Those are given by sections of p, say $p_i : S \to C$, which are distinct points on the geometric fibers. Then the morphisms are diagrams

$$\begin{tikzcd} C' \arrow[r, "f"] \arrow[d, shift left] & C \arrow[d, shift right] \\ S' \arrow[r, "g"] \arrow[u, bend left, "p'_1,\ldots,p'_n"] & S \arrow[u, bend right, "p_1,\ldots,p_n"'] \end{tikzcd}$$

such that $\pi f = g\pi'$ and $fp'_i = p_i g$. Then, the projection $p : \mathcal{M}_{g,n} \to (\mathbf{Sch})$ is given by $(C \xrightarrow{\pi} S) \mapsto S$.

One should check that pullbacks exist.

In particular, $\mathcal{M}_{1,1}$ is the moduli space of genus 1 curves with one marked point, i.e. they are exactly elliptic curves.

Next, note we have the fibered category $(\mathbf{QCoh}) \to (\mathbf{Sch})$ and we get $F_i : \mathcal{M}_{g,n} \to (\mathbf{QCoh})$ map of fibered category, for $1 \le i \le n$. The map is given by

$$F_i(C \xrightarrow{\pi} S) := (S, p_i^*\Omega^1_{C/S})$$

where we take the pullback of relative differential via p_i. We also need to define what F_i does on morphisms. Well, suppose we have morphism

$$\begin{tikzcd} C' \arrow[r, "f"] \arrow[d, shift left] & C \arrow[d, shift right] \\ S' \arrow[r, "g"] \arrow[u, bend left, "p'_1,\ldots,p'_n"] & S \arrow[u, bend right, "p_1,\ldots,p_n"'] \end{tikzcd}$$

We need a map between $(p'_i)^*\Omega^1_{C'/S'} \to g^*P_i^*\Omega^1_{C/S}$. Well, we do have a canonical morphism (and in fact its isomorphism), as we will show next. First, note $g^*p_i^*\Omega^1_{C/S}$ is equal to $(p'_i)^*f^*\Omega^1_{C/S}$ as the diagram commutes, and we also have $(p'_i)^*f^*\Omega^1_{C/S} = (p'_i)^*\Omega^1_{C'/S'}$. Hence we get the desired canonical (iso)morphism as desired.

Lemma 2.1.14

Suppose $g : \mathcal{F} \to \mathcal{G}$ is a map of fibered category. Then g is fully faithful as a map of categories (not fibered category) if and only if $\forall U \in \mathcal{C}$, $g_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is fully faithful.

This result should remind you of: if $\mathcal{F} \to \mathcal{G}$ is a map of presheaves then it is injective if and only if for all U, $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective.

Proof. Recall fully faithful means we have a bijection between Hom sets (full means surjection between hom sets and faithful means injection between hom sets). Thus, let $x, y \in \mathcal{F}$, we get

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{F}}(x,y) \arrow[r, "g"] \arrow[dr, "p_{\mathcal{F}}"] & \mathrm{Hom}_{\mathcal{G}}(g(x), g(y)) \arrow[d, "p_{\mathcal{G}}"] \\ & \mathrm{Hom}_{\mathcal{C}}(p_{\mathcal{F}}(x), p_{\mathcal{F}}(y)) \end{tikzcd}$$

Then g is fully faithful if and only if for all $h : p_{\mathcal{F}}(x) \to p_{\mathcal{F}}(y)$ in $\mathrm{Hom}_{\mathcal{C}}(p_{\mathcal{F}}(x), p_{\mathcal{F}}(y))$, g induces a bijection

$$\{x \xrightarrow{\phi} y : p_{\mathcal{F}}(\phi) = h\} \xrightarrow{\sim} \{g(x) \xrightarrow{\psi} g(y) : p_{\mathcal{G}}(\psi) = h\}$$

This is sort of like we show bijection on each of the fibers.

Thus, we fix h downstairs

$$\begin{tikzcd} & y \arrow[d] \\ p_{\mathcal{F}}(x) \arrow[r, "h"] & p_{\mathcal{F}}(y) \end{tikzcd}$$

and let y' be a fibered product/pullback

$$\begin{tikzcd} y' \arrow[r] \arrow[d, "\tilde{h}"] \arrow[dr, phantom, "\square"] & y \arrow[d] \\ p_{\mathcal{F}}(x) \arrow[r, "h"] & p_{\mathcal{F}}(y) \end{tikzcd}$$

Then we see for all $x \to y$, we get unique arrow $x \to y'$, i.e. we get

$$\begin{tikzcd} x \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr] & & \\ & y' \arrow[r, "\tilde{h}"'] \arrow[d] \arrow[dr, phantom, "\square"] & y \arrow[d] \\ & p_{\mathcal{F}}(x) \arrow[r, "h"] & p_{\mathcal{F}}(y) \end{tikzcd}$$

so, we have a bijection

$$\{x \xrightarrow{\phi} y : p_{\mathcal{F}}(\phi) = h\} = \{x \xrightarrow{\phi'} y' : p_{\mathcal{F}}(\phi') = \mathrm{Id}\}$$

This is because when we actually using definition of pullback, what we get is a diagram

$$\begin{tikzcd} x \arrow[r, dashed, "\exists!\phi'"] \arrow[rr, bend left, "\forall\phi"] \arrow[d] & y' \arrow[r, "\tilde{h}"'] \arrow[d] & y \arrow[d] \\ p_{\mathcal{F}}(x) \arrow[r, "\mathrm{Id}"'] & p_{\mathcal{F}}(x) \arrow[r, "h"'] & p_{\mathcal{F}}(y) \end{tikzcd}$$

Now, since $g(\tilde{h})$ is Cartesian arrow as $\tilde{h}$ is Cartesian (by def of morphism of fibered cat), we see we get

$$\begin{tikzcd} g(y') \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & g(y) \arrow[d] \\ p_{\mathcal{F}}(x) \arrow[r, "h"] & p_{\mathcal{F}}(y) \end{tikzcd}$$

We also have the $g(x)$ floating around, and we get

$$\begin{tikzcd}
g(x) \arrow[drr] \arrow[dr, dotted, "\exists!"] \arrow[ddr] & & \\
& g(y') \arrow[r, "g(\tilde{h})"'] \arrow[d] \arrow[dr, phantom, "\square"] & g(y) \arrow[d] \\
& p_{\mathcal{F}}(x) \arrow[r] & p_{\mathcal{F}}(y)
\end{tikzcd}$$

Hence we get bijection

$$\{g(x) \to g(y) \text{ lying over } h\} = \{g(x) \to g(y') \text{ lying over Id}\}$$

with the similar reasoning as above. Hence,

$$\{x \to y \text{ lying over } h\} \xrightarrow{g} \{g(x) \to g(y) \text{ lying over } h\}$$

is a bijection if and only if $g_{p_{\mathcal{F}}(x)} : \mathcal{F}(p_{\mathcal{F}}(x)) \to \mathcal{G}(p_{\mathcal{F}}(x))$ is fully faithful. This concludes the proof.

Definition 2.1.15

We say $g : \mathcal{F} \to \mathcal{G}$ a map of fibered categories is an ***equivalence*** if there exists $h : \mathcal{G} \to \mathcal{F}$ map of fibered categories and exists a base preserving isomorphism $\alpha : g \circ h \xrightarrow{\sim} \mathrm{Id}_{\mathcal{G}}$ and $\beta : h \circ g \xrightarrow{\sim} \mathrm{Id}_{\mathcal{F}}$.

Proposition 2.1.16

For a map of fibered categories $g : \mathcal{F} \to \mathcal{G}$, g is equivalence iff $\forall U \in \mathcal{C}$, g_U is an equivalence (in category theory sense) iff $\forall U \in \mathcal{C}$, g_U is fully faithful and essentially surjective.

Proof. We already showed g is fully faithful if and only if all g_U are. Thus we just need to show the claims about essentially surjective (recall essentially surjective for $g : \mathcal{F} \to \mathcal{G}$ means each object $y \in \mathcal{G}$ is isomorphic to an object of the form $g(x)$ where $x \in \mathcal{F}$).

($\Rightarrow$) : if g is equivalence, then we want g_U to be essentially surjective. Given $y \in \mathcal{G}(U)$, we have $gh(y) \xrightarrow{\sim} y$ and h is a morphism of fibered cats[3], so $h(y) \in \mathcal{F}(U)$.

($\Leftarrow$): now assume g_U is essentially surjective for all U. We need to construct an equivalence of fibered categories $h : \mathcal{G} \to \mathcal{F}$. Given $y \in \mathcal{G}(U)$, since g_U is equivalence,

[3] from time to time we will write cat to mean category

we know there exists $h(y) \in \mathcal{F}(U)$ such that $\alpha_y : y \xrightarrow{\sim} g_U(h(y))$. Given any $y \xrightarrow{\phi} y'$ in $\mathcal{G}(U)$, there exists unique $h(\phi) : h(y) \to h(y')$ in $\mathcal{F}(U)$ such that

$$\begin{tikzcd}
y \arrow[r, "\phi"] \arrow[d, "\alpha_y", "\sim"'] & y' \arrow[d, "\alpha_{y'}", "\sim"'] \\
g_U(h(y)) \arrow[r, "g_U(h(\phi))"] & g_U(h(y'))
\end{tikzcd}$$

because g_U is fully faithful.

This gives functor $h : \mathcal{G} \to \mathcal{F}$ and also $\alpha : \mathrm{Id}_{\mathcal{G}} \xrightarrow{\sim} g \circ h$. We need h sends Cartesian arrows to Cartesian arrows. If $y \xrightarrow{\phi} y'$ is Cartesian in $\mathcal{G}$, then we get

$$\begin{tikzcd}
h(y) \arrow[r] \arrow[d] & h(y') \arrow[d] \\
p_{\mathcal{F}}(h(y)) \arrow[r] & p_{\mathcal{F}}(h(y'))
\end{tikzcd}$$

and suppose we are given arbitrary w with the following diagram

$$\begin{tikzcd}
w \arrow[rr, bend left] \arrow[d] & h(y) \arrow[r] \arrow[d] & h(y') \arrow[d] \\
p_{\mathcal{F}}(w) \arrow[r] & p_{\mathcal{F}}(h(y)) \arrow[r] & p_{\mathcal{F}}(h(y'))
\end{tikzcd}$$

where we want to show there exists unique arrow $w \to h(y)$. Since g is fully faithful, there exists dotted arrow $w \to h(y)$ if and only if it holds after we apply g. Thus we want to show there exists unique dotted arrow in the following:

$$\begin{tikzcd}
g(w) \arrow[r, dotted, "\exists!"] \arrow[rr, bend left] \arrow[d] & gh(y) \arrow[r] \arrow[d] & gh(y') \arrow[d] \\
p_{\mathcal{F}}(g(w)) \arrow[r] & p_{\mathcal{F}}(gh(y)) \arrow[r] & p_{\mathcal{F}}(gh(y'))
\end{tikzcd}$$

However, note we can complete the diagram with

$$\begin{tikzcd}
y \arrow[r, "\phi"] \arrow[d, "\alpha_y", "\sim"'] & y' \arrow[d, "\alpha_{y'}", "\sim"'] \\
gh(y) \arrow[r] & gh(y')
\end{tikzcd}$$

In other word, we get

$$\begin{tikzcd}
 & y \arrow[r] \arrow[d] & y' \arrow[d] \\
g(w) \arrow[r, dotted] \arrow[d] \arrow[urr, bend left=60] & gh(y) \arrow[r] \arrow[d] & gh(y') \arrow[d] \\
p_{\mathcal{F}}(g(w)) \arrow[r] & p_{\mathcal{F}}(gh(y)) \arrow[r] & p_{\mathcal{F}}(gh(y'))
\end{tikzcd}$$

but $y \to y'$ is Cartesian, hence we indeed have the dotted arrow as desired.

Lastly, we need $\beta : \mathrm{Id}_{\mathcal{F}} \xrightarrow{\sim} h \circ g$. If $x \in \mathcal{F}$, we want $x \xrightarrow[\beta_x]{\sim} h(g(x))$. By full faithfulness of g, we just need to show $y(x) \xrightarrow{\sim} g(h(g(x)))$. But we do have an isomorphism, $\alpha_{g(x)}$. Then, we left as an exercise that β_x is a natural transformation.

Now we have seen fibered categories are analogous to presheaves (over (**Sets**)), we ask a natural question: what is analogue of sheaf? The answer is stacks.

But to be able to talk about this, recall in the ordinary case, we need Yoneda lemma. But our categories are 2-categories, so we really need to ask, is there a type of Yoneda lemma in this case? Well, there is, and its called 2-Yoneda lemma.

Before we do this, let's recall if $X \in \mathcal{C}$, we have fibered category maps $\mathcal{C}/X \to \mathcal{C}$ with morphism $(Y \to X) \mapsto Y$. The analogy is that, $\mathcal{C}/X$ should correspond to h_X.

Theorem 2.1.17: 2-Yoneda Lemma

For any fibered category $\mathcal{F} \to \mathcal{C}$, and all $X \in \mathcal{C}$, we have a category $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F})$ *with morphisms being base-preserving natural transformations. Then, we have a equivalence of categories*

$$\begin{aligned} \zeta : \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F}) &\to \mathcal{F}(X) \\ g &\mapsto g(X \xrightarrow{\mathrm{Id}_X} X) \end{aligned}$$

Proof. We need to construct $\eta : \mathcal{F}(X) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F})$ that maps $x \in \mathcal{F}(X)$ to a fibered category $(\eta_x : \mathcal{C}/X \to \mathcal{F})$.

First, we define η_x on objects: an object in $\mathcal{C}/X$ is a map $Y \xrightarrow{\phi} X$. In particular, we have diagram

$$\begin{tikzcd}
 & x \arrow[d, mapsto] & \mathcal{F} \arrow[d] \\
Y \arrow[r, "\phi"] & X & \mathcal{C}
\end{tikzcd}$$

Well, the natural thing to do is just take a pullback ϕ^*x, i.e. we make a choice and get the following diagram

$$\begin{tikzcd}
\phi^*x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[d, mapsto] \\
Y \arrow[r, "\phi"] & X
\end{tikzcd}$$

and define $\eta_x(\phi) := \phi^*x \in \mathcal{F}(Y)$.

Next, we define η_x on morphisms. Suppose we are given morphism

$$\begin{tikzcd}
Y' \arrow[r, "\xi"] \arrow[dr, "\phi'"] & Y \arrow[d, "\phi"] \\
& X
\end{tikzcd}$$

in $\mathcal{C}$. We want to know what $\eta_x(\xi)$ is. Well, we get the following diagram

$$\begin{tikzcd}
(\phi')^*x \arrow[dd] \arrow[drr, bend left] & & \\
& \phi^*x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[d] \\
Y' \arrow[r] & Y \arrow[r] & X
\end{tikzcd}$$

but then we get a dotted arrow between $(\phi')^*x$ to ϕ^*x as the squares are Cartesian. Hence we have

$$\begin{tikzcd}
(\phi')^*x \arrow[dd] \arrow[drr, bend left] \arrow[dr, dotted, "\exists!"] & & \\
& \phi^*x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[d] \\
Y' \arrow[r] & Y \arrow[r] & X
\end{tikzcd}$$

This unique dotted arrow gives the desired map on morphisms (i.e. $\eta_x(\xi) : (\phi')^*x \to \phi^*x$).

Now we know η as a functor $\mathcal{F} \to \mathcal{C}$, why is η a morphism of fibered categories. We have to check two things: the first thing is that it respect fibers. However, we checked that already, i.e. $(Y \xrightarrow{\phi} X) \mapsto$ something in $\mathcal{F}(Y)$.

The second thing we need to show is that η takes Cartesian arrows to Cartesian. First, in $\mathcal{C}/X$ all arrows are Cartesian, so we need $\eta_x(\text{any arrow}) = \text{Cartesian}$. However, note by basic category theory, since the inner square and outer squares are both Cartesian, the dotted arrow must also be Cartesian.

So, we now know η on objects. What about morphisms?

Given $f : x' \to x$ in $\mathcal{F}(X)$, we want $\eta_f : \eta_{x'} \to \eta_x$, i.e. we need η_f to be base-preserving natural trans.

Given $\phi : Y \to X$ in $\mathcal{C}/X$, we need $\eta_f(\phi) : \eta_{x'}(\phi) = \phi^* x' \to \eta_x(\phi) = \phi^* x$. Lets draw out the diagrams

$$\begin{tikzcd}
(\phi)^*x' \arrow[rr] \arrow[ddr] & & x' \arrow[d, "f"'] \arrow[dd, bend left] \\
& \phi^*x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[d] \\
& Y \arrow[r, "\phi"] & X
\end{tikzcd}$$

but then we get a unique dotted arrow

$$\begin{tikzcd}
(\phi)^*x' \arrow[rr] \arrow[ddr] \arrow[dr, dotted, "\exists!"] & & x' \arrow[d, "f"'] \arrow[dd, bend left] \\
& \phi^*x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[d] \\
& Y \arrow[r, "\phi"] & X
\end{tikzcd}$$

this is our definition of $\eta_f(\phi)$. We still need to show this is base-preserving, i.e. it lives over the identity. Indeed, note $f : x' \to x$ lives over the identity (as f is a morphism in $\mathcal{F}(X)$), we see the pullback, i.e. the dotted arrow, also lives over identity. Hence η_f is base-preserving natural transformation as desired.

We have now defined η. Next we need to show this is an equivalence.

First, we show $\zeta\eta = \mathrm{Id}$. Note we have

$$\zeta\eta(x) = \zeta(\eta_x) = \eta_x(X \xrightarrow{\mathrm{Id}_X} X)$$

but note

$$\eta_x(\mathrm{Id}_x) : \mathcal{C}/X \to \mathcal{F}(X)$$

is given by the following square

$$\begin{tikzcd}
(\mathrm{Id}_X)^*x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[d] \\
X \arrow[r, "\mathrm{Id}_X"] & X
\end{tikzcd}$$

but $(\mathrm{Id}_X)^*x = x$ and hence $\zeta\eta(x) = x$ as desired.

Next, we show $\eta\zeta \cong \mathrm{Id}$. We see we get

$$\eta\zeta(f : \mathcal{C}/X \to \mathcal{F}) = \eta(f(\mathrm{Id}_X)) = \eta_{f(\mathrm{Id}_X)}$$

where we see

$$\eta_{f(\mathrm{Id}_X)} : \mathcal{C}/X \to \mathcal{F}$$

and by definition we get the following diagram

$$\begin{tikzcd}
& f(\mathrm{Id}_X) \arrow[d] \\
Y \arrow[r, "\phi"] & X
\end{tikzcd}$$

and taking pullback we get

$$\begin{tikzcd} \phi^*(f(\mathrm{Id}_X)) = \eta_{f(\mathrm{Id}_X)}(\phi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & f(\mathrm{Id}_X) \arrow[d] \\ Y \arrow[r, "\phi"] & X \end{tikzcd}$$

and we want to show $\eta_{f(\mathrm{Id}_X)}(\phi) = \phi^*(f(\mathrm{Id}_X)) \cong f(\phi)$ because then $\eta_{f(\mathrm{Id}_X)}(\phi) = f(\phi)$, i.e. $\eta\zeta(f) \cong f$.

To show this, it is enough to show the arrow $f(\phi) \to f(\mathrm{Id}_X)$ is Cartesian because if we have Cartesian diagrams

$$\begin{tikzcd} f(\phi) \arrow[drr] \arrow[ddr] \arrow[dr, phantom, "\square"] & & \\ & \phi^*(f(\mathrm{Id}_X)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & f(\mathrm{Id}_X) \arrow[d] \\ & Y \arrow[r] & X \end{tikzcd}$$

then since 2 pullbacks are canonically isomorphic we get the desired isomorphism.

Now, why is the arrow $f(\phi) \to f(\mathrm{Id}_X)$ Cartesian? We always have unique canonical map $\phi \to \mathrm{Id}_X$, as recall morphism of morpihsmes in our case is just try to fill the following diagram:

$$\begin{tikzcd} Y \arrow[dr, "\phi"] & X \arrow[d, "\mathrm{Id}_X"] \\ & X \end{tikzcd}$$

but there is only a unique way to do this, which is

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[dr, "\phi"] & X \arrow[d, "\mathrm{Id}_X"] \\ & X \end{tikzcd}$$

This gives an arrow $\phi \to \mathrm{Id}_X$ which is automatically Cartesian. Thus $f(\phi) \to f(\mathrm{Id}_X)$ is Cartesian because f preserves Cartesian arrows. This concludes the proof.

Corollary 2.1.17.1

For $X, Y \in \mathcal{C}$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{C}/Y) &\to \mathrm{Hom}_{\mathcal{C}}(X, Y) \\ f &\mapsto f(\mathrm{Id}_X) \end{aligned}$$

is an equivalence of categories.

Note in the above, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a set viewed as a category with objects equal set elements, and morphisms being only identity maps.

Proof. Well, apply 2-Yoneda lemma to $\mathcal{F} = \mathcal{C}/Y$, we get

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{C}/Y) = (\mathcal{C}/Y)(X)$$

where the objects are $X \xrightarrow{\phi} Y$ and morphisms are

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[dr, "\phi"] & X \arrow[d, "\phi'"] \\
& Y
\end{tikzcd}$$

living over Id_X, i.e. $\psi = \mathrm{Id}_X$. Thus, the only morphism we have in $(\mathcal{C}/Y)(X)$ are identity maps, i.e. $(\mathcal{C}/Y)(X)$ is exactly the category $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.

Thus, we will introduce some notations: we will frequently write $X \to \mathcal{F}$ in place of $\mathcal{F}(X)$ if $\mathcal{F}$ is fibered category. This is justified by 2-Yoneda because $\mathcal{C}/X \to \mathcal{F}$ is the same as $\mathcal{F}(X)$. So it is just a convenience to write X in place of $\mathcal{C}/X$. The corollary shows $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{C}/Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$, so $X \to Y$ in place of $\mathcal{C}/X \to \mathcal{C}/Y$ is unambiguous.

2.2 Category Fibered In Groupoids

Definition 2.2.1

A ***caregory fibered in sets over*** $\mathcal{C}$ is a fibered category $\mathcal{F} \to \mathcal{C}$ such that for all $U \in \mathcal{C}$, $\mathcal{F}(U)$ is a set, i.e. only maps are identity.

We note in category theory, a set means a category where the only maps are identity (this is definition).

Note for such $\mathcal{F}$, we have a well-defined pullback map. Indeed, we get diagram

$$\begin{tikzcd}
y' \arrow[drr] \arrow[ddr] \arrow[dr, phantom, "\square"] & & \\
& y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[d] \\
& U \arrow[r] & V
\end{tikzcd}$$

Then we get $y' \to y$ lying over Id_U, i.e. $y' \to y$ is morphism in $\mathcal{F}(U)$. But $\mathcal{F}(U)$ is a set, so it must be identity by definition, i.e. $y' = y$.

So, given $U \xrightarrow{f} V$, we get $f^* : \mathcal{F}(V) \to \mathcal{F}(U)$ compatible with composition, i.e. $\mathcal{F}$ naturally yields a presheaf $F_{\mathcal{F}}$ where $F_{\mathcal{F}}(U) := \mathcal{F}(U)$.

The next task is to show categories fibered in sets are the same as presheaves.

Lemma 2.2.2

If $\mathcal{F} \to \mathcal{C}$ is a category fibered in sets and $\mathcal{G} \to \mathcal{C}$ is any fibered category, then $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, \mathcal{F})$ is a set.

We note again that in higher category theory, sets are, by definition, categories with only identity morphism.

Proof. For $f, g : \mathcal{G} \to \mathcal{F}$ morphisms of fibered category, and $\alpha : f \to g$ morphism, we want to show $\alpha = \mathrm{Id}$. For all $x \in \mathcal{G}(X)$, $\alpha_x : f(x) \to g(x)$ is in $\mathcal{F}(X)$, so $\alpha_x = \mathrm{Id}$. Also, given $\phi : y \to x$, we get diagram

$$\begin{tikzcd} f(y) \arrow[r, "f(\phi)"] \arrow[d, "\alpha_y", "="'] & f(x) \arrow[d, "\alpha_x", "="'] \\ g(y) \arrow[r, "g(\phi)"] & g(x) \end{tikzcd}$$

and hence $f(\phi) = g(\phi)$. Hence $f = g$ and α is Id_f.

Corollary 2.2.2.1

Categories fibered in sets over $\mathcal{C}$ is a (locally small) category, i.e. $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, \mathcal{F})$ is a set.

Example 2.2.3

Given a presheaf $F : \mathcal{C}^{\mathrm{opp}} \to (\mathbf{Sets})$, let $\mathcal{F} := \mathcal{F}_F$ be the following category: objects are (U, u) with $U \in \mathcal{C}$, $u \in F(U)$, morphisms $(U', u') \to (U, u)$ are $g : U' \to U$ in $\mathcal{C}$ such that $g^* : F(U) \to F(U')$ so $u \mapsto u'$.

This is a fibered category because: we just let $p : \mathcal{F} \to \mathcal{C}$ be $(U, x) \mapsto U$, and we need to show pullback exists. Suppose we have

$$\begin{tikzcd} g^*x \arrow[r] \arrow[d, "p"] & x \arrow[d, "p"] \\ U' \arrow[r, "g"] & U \end{tikzcd}$$

we claim this is a pullback, i.e. all maps are Cartesian.

To check this, we have the following diagram

$$\begin{tikzcd} U'' \arrow[r, "f"] \arrow[rr, bend right, "h"'] & U' \arrow[r, "g"] & U \end{tikzcd}$$

In this diagram, we get $x \mapsto U, g^*x \mapsto U'$ and suppose we are given h^*x,

$$\begin{tikzcd} h^*x \arrow[drr, bend left] & & \\ & g^*x \arrow[r] & x \\ U'' \arrow[r, "f"] \arrow[rr, bend right, "h"'] & U' \arrow[r, "g"] & U \end{tikzcd}$$

We are trying to show there is unique map $h^*x \to g^*x$, i.e. we want

$$\begin{tikzcd} h^*x \arrow[drr, bend left] \arrow[dr, dotted, "\exists!"] & & \\ & g^*x \arrow[r] & x \\ U'' \arrow[r, "f"] \arrow[rr, bend right, "h"'] & U' \arrow[r, "g"] & U \end{tikzcd}$$

Why this map exists? By definition, $h^*x \to x$ means $(gf)^*x = f^*g^*x = f^*(g^*x)$ as we are working with presheaf. Hence we indeed get the desired arrow $h^*x \to g^*x$.

We should also check $\mathcal{F}(U)$ is a set, so that $\mathcal{F} \to \mathcal{C}$ is fibered in sets.

The objects of $\mathcal{F}(U)$ are (U,x) with $x \in F(U)$. The morphisms are $(U,y) \to (U,x)$ lying over identity Id_U, i.e.

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x \arrow[d] \\ U \arrow[r, "\mathrm{Id}_U"] & U \end{tikzcd}$$

but $y = \mathrm{Id}_U^*(x)$ and hence $y = x$. Thus morphisms in $\mathcal{F}(U)$ are identity, i.e. $\mathcal{F}(U)$ is a set.

Proposition 2.2.4

There is an equivalence of categories between

$$\left\{\begin{matrix}\textit{presheaves on}\\ \mathcal{C}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{categories fibered}\\ \textit{in sets over } \mathcal{C}\end{matrix}\right\}$$

given by

$$F \mapsto \mathcal{F}_F$$
$$F_{\mathcal{F}} \leftarrow\!\shortmid \mathcal{F}$$

We defined the maps already, so one should check these are quasi-inverse maps

Throughout the course, we will identify $\mathcal{F}$ with $F_{\mathcal{F}}$.

Definition 2.2.5

A category is a ***groupoid*** if all morphisms are isomorphisms.

Example 2.2.6

If G is a group, then the category with one object and morphisms being G is a groupoids (i.e. if object is $\bullet$, then we get an arrow $\bullet \xrightarrow{g} \bullet$ for each $g \in G$). Thus groups are examples of groupoids. In general a finite groupoid can have a lot of dots, and all the arrows are isomorphism, i.e. those can be viewed as generalizations of groups.

Example 2.2.7

Now let's see some examples of groupoids.

- Let X be a topological space (not necessarily path-connected). Consider the category $\Pi_1(X)$ with $\mathrm{Obj}(\Pi_1) = X$, and $\mathrm{Hom}(x, y)$ the homotopy equivalence classes of continuous paths from x to y. The composition is given by follow the path. In particular, one checks $\mathrm{Hom}(x, x) = \pi_1(X, x)$, the fundamental group of X at x.
- Let G be a group acting on a set X, then we can define the action groupoid as follows:
  - The objects are the elements of X.
  - For any two elements x, y in X, the morphisms from x to y is equal $\{g \in G : gx = y\}$.
  - Composition of morphisms is just the multiplication.

  In this case, $\mathrm{Hom}(x, x)$ is exactly the isotropy group of x.

Definition 2.2.8

A ***category fibered in groupoids*** is a fibered category $\mathcal{F} \to \mathcal{C}$ such that all $\mathcal{F}(U)$ are groupoids.

Before we start investigate this type of category, let's give a little bit motivation. Recall one of the end game for us is to define and study moduli spaces. In particular, moduli spaces are roughly "objects"/"isomorphisms". The notion of category fibered in groupoid is then motivated from the need of a more sophisticated definition of quotients.

Indeed, let's start with a set X and equivalence relation $R \subseteq X \times X$. Then this information can be recorded via the diagram

$$\begin{tikzcd} R \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_2"'] & X \arrow[l, "\Delta" description] \end{tikzcd}$$

where p_1, p_2 are the projections from R to X, and $\Delta : X \to R$ is the diagonal map $\Delta(x) = (x, x)$. This diagram defines a groupoid $[X/R]$:

- The objects of this category $[X/R]$ are elements of X
- The morphisms of $[X/R]$ are the elements of R
- The "source" and "target" maps $\mathrm{Mor}([X/R]) \to \mathrm{Obj}([X/R])$ are given by the projections p_1, p_2
- The "identity" map $\mathrm{Obj}([X/R]) \to \mathrm{Mor}([X/R])$ is given by the diagonal s
- Composition of morphisms is well-defined as relation is transitive
- All morphisms are invertible since the relation is symmetric

Unlike the set quotient X/R, the groupoid $[X/R]$ remembers how elements are identified (i.e. in X/R we can only tell when $[x] = [y]$, while in $[X/R]$ we can also tell how $[x] = [y]$).

This can be extended to define $[X/G]$ where G is a group acting on X. This is given by the diagram

$$\begin{tikzcd} G \times X \arrow[r, shift left=2, "p_2"] \arrow[r, shift right=2, "\text{action}"'] & X \arrow[l, "s" description] \end{tikzcd}$$

where the "source" map is the second projection p_2, the "target" map is the action map $(g, x) \mapsto gx$, and the "identity" is $x \mapsto (e, x)$. In this case, X/G only contains information of whether x, y lies in the same orbit, while the groupoid $[X/G]$ contains one isomorphism $x \cong y$ for every $g \in G$ such that $g \cdot x = y$. This suggests instead of just sets, we should allow moduli spaces to have groupoids of points.

Proposition 2.2.9

If $\mathcal{F}, \mathcal{F}'$ are categories fibered in groupoids over $\mathcal{C}$, then the category $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{F}')$ *is a groupoid.*

Proof. Let $f, g : \mathcal{F} \to \mathcal{F}'$ with $\xi : f \to g$. We need to show ξ is isomorphism, i.e. forall $x \in \mathcal{F}$, we need to show $\xi_x : f(x) \xrightarrow{\sim} g(x)$. Let $X = p_{\mathcal{F}}(x)$, then since ξ is base-preserving natural trans, then ξ_x lies over Id_X, i.e. $\xi_x \in \mathcal{F}'(X) =$ groupoid, i.e. ξ_x is isomorphism.

We note in the proof, we only need $\mathcal{F}'$ being fibered in groupoids. Also, by the same argument, one can show that if $\mathcal{F} \to \mathcal{C}$ is category fibered in groupoids, then all arrows are Cartesian.

Next we consider a nice example of categories fibered in groupoids.

Definition 2.2.10

A ***groupoid in*** $\mathcal{C}$ is $(X_0, X_1, s, t, \epsilon, i, m)$ such that $X_0, X_1 \in \mathcal{C}$ and

$$\begin{tikzcd} X_1 \arrow[loop left, "i"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & X_0 \arrow[l, bend left, "\epsilon"] \end{tikzcd}$$

and we get Cartesian square

$$\begin{tikzcd} X_1 & & \\ & X_1 \times_{s, X_0, t} X_1 \arrow[ul, "m"'] \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\square"] & X_1 \arrow[d, "t"] \\ & X_1 \arrow[r, "s"] & X_0 \end{tikzcd}$$

We note in the above, we used the notation $X_1 \times_{s, X_0, t} X_1$, which is just the fibered product, but we put emphasis on the two maps s and t that defines the fibered product.

So, here is the intuition: Here:

1. s: source
2. t: target
3. ϵ: identity
4. i: inverse
5. m: multiplication/composition

Here is how you supposed to think of this set of data.

X_0 is supposed to be like objects in a category, X_1 is supposed to be arrows in the category. Then what's going on with s and t in

$$\begin{tikzcd} X_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & X_0 \end{tikzcd}$$

is that, s takes an arrow and sends it to the source, while t takes an arrow and sends it to the target.

$\epsilon : X_0 \to X_1$ takes objects to the identity arrow (so if $x \in X_0$ is our "object", then $\epsilon(x)$ should be thought as $\mathrm{Id}_x \in \mathrm{Hom}(x, x)$).

$i : X_1 \to X_1$ takes an arrow to its inverse (this supposed to exists because its a groupoid).

$m : X_1 \times_{s, X_0, t} X_1 \to X_1$ is like an arrow (α, β) such that the source of α is equal the target of β. In other word, (α, β) under m is supposed to be "$\alpha \circ \beta$".

The above is the intuition, and let's give the actual axioms about groupoids in $\mathcal{C}$.

Axioms:

1. $s \circ \epsilon = \mathrm{Id} = t \circ \epsilon$
2. $t \circ i = s$ and $s \circ i = t$.
3. $s \circ m = s \circ p_2$
4. $t \circ m = t \circ p_1$
5. (Associativity): the following two maps (we dropped the s, X_0, t in all the fibered products here)

$$X_1 \times X_1 \times X_1 \underset{\mathrm{Id}\times m}{\overset{m\times \mathrm{Id}}{\rightrightarrows}} X_1 \times X_1 \xrightarrow{m} X_1$$

   are equal.
6. (Identity):

$$\begin{tikzcd}
 & X_1 \times_{s,X_0,\mathrm{Id}} X_0 \arrow[dr, "\epsilon\times\mathrm{Id}"] & & \\
X_1 \arrow[ur, "="] \arrow[dr, "="'] & & X_1 \times_{s,X_0,t} X_1 \arrow[r, "m"] & X_1 \\
 & X_0 \times_{\mathrm{Id},X_0,t} X_1 \arrow[ur, "\mathrm{Id}\times\epsilon"'] & &
\end{tikzcd}$$

   This says $\alpha \circ \mathrm{Id} = \alpha$ and $\mathrm{Id} \circ \alpha = \alpha$.
7. (Inverse): we get diagrams

$$\begin{tikzcd}
X_1 \arrow[r, "i\times\mathrm{Id}"] \arrow[d, "s"] & X_1 \times_{s,X_0,t} X_1 \arrow[d, "m"] \\
X_0 \arrow[r, "\epsilon"] & X_1
\end{tikzcd}$$

$$\begin{tikzcd}
X_1 \arrow[r, "\mathrm{Id}\times i"] \arrow[d, "t"] & X_1 \times_{s,X_0,t} X_1 \arrow[d, "m"] \\
X_0 \arrow[r, "\epsilon"] & X_1
\end{tikzcd}$$

   This says that $\alpha^{-1} \circ \alpha = \mathrm{Id}$ and $\alpha \circ \alpha^{-1} = \mathrm{Id}$.

In the above we listed the axioms of groupoids in $\mathcal{C}$. The definition seems complicated, but the idea is not bad. Basically, X_0 should be objects, X_1 be morphisms, s takes morpihsms to its source, and t to the target, ϵ sends objects to identity map, i to inverse arrow, and m is composition of arrows.

Next we are going to show we can get from groupoids in $\mathcal{C}$ to categories fibered in groupoids over $\mathcal{C}$.

Given $U \in \mathcal{C}$, let $\{X_0(U)/X_1(U)\}$ be a category defined as follows: objects are $X_0(U) = \mathrm{Hom}_{\mathcal{C}}(U, X_0)$, and morphisms for $u \to u'$ is an element $\alpha \in X_1(U)$ such that

$s(\alpha) = u$ and $t(\alpha) = u'$ (here $u, u' : U \to X_0$, $\alpha : U \to X_1$ and hence $s(\alpha) = s \circ \alpha$ is an arrow $U \to X_0$, i.e. it make sense to ask $s(\alpha) = u$ and so on). This is a category, where composition of arrows are given by apply m, i.e. say we have $\eta : u'' \to u'$ and $\xi : u' \to u$, where $u'' \to u$ is given by $m(\xi, \eta)$.

Next, we define a fibered category $\mathcal{F} = \{X_0/X_1\}$ over $\mathcal{C}$ as follows: objects are (U, u), where $U \in \mathcal{C}$ and $u \in \{X_0(U)/X_1(U)\}$ (recall objects of this category is just $X_0(U)$). To get the morphisms, note given $f : V \to U$, we get the arrow (which is a functor)

$$\{X_0(U)/X_1(U)\} \xrightarrow{f^*} \{X_0(V)/X_1(V)\}$$

is well-defined (i.e. $f : V \to U$ induces $f^* : X_0(U) \to X_0(V)$ and $f^* : X_1(U) \to X_1(V)$ and hence $f^* : \{X_0(U)/X_1(U)\} \to \{X_0(V)/X_1(V)\}$). Then, morphisms $(V, v) \to (U, u)$ will be given by pairs $f : V \to U$ and $\alpha : v \xrightarrow{\sim} f^*u$ an isomorphism in $\{X_0(V)/X_1(V)\}$.

Then the projection $p : \mathcal{F} \to \mathcal{C}$ is going to be $p(U, u) = U$.

It remains to check $p : \mathcal{F} \to \mathcal{C}$ is a category fibered in groupoids.

The fiber $\mathcal{F}(U)$ is the category defined by: objects are, by definition, just $X_0(U)$ (as it is the same as the objects of $\{X_0(U)/X_1(U)\}$). The morphism for $(U, v) \to (U, u)$ is given by $U \xrightarrow{f} U$ and $\alpha : v \xrightarrow{\sim} f^*u$. However, since f must live over the identity, $f = \mathrm{Id}$ and hence $f^*u = u$. In other word, morphisms are just $X_1(U)$, i.e. $\mathcal{F}(U) = \{X_0(U)/X_1(U)\}$ is a groupoid, as desired.

Aside, if $\mathcal{F} \xrightarrow{p} \mathcal{C}$ is category fibered in groupoids, for $X \in \mathcal{C}$ we can define $p/X : \mathcal{F}/X \to \mathcal{C}/X$, which is a category fibered in groupoids where $\mathcal{F}/X$ behaves like objects of $\mathcal{F}$ over X. This notion is hardly been used, so if we need it in the future we will define it, but for now its just aside.

The next notion is rather important.

Definition 2.2.11

Given $\mathcal{F} \xrightarrow{p} \mathcal{C}$ a category fibered in groupoids, $x, x' \in \mathcal{F}(X)$. We define a preasheaf

$$\mathrm{Isom}(x, x') : (\mathcal{C}/X)^{\mathrm{opp}} \to (\mathbf{Sets})$$

as follows.

Let $f : Y \to X$ in $\mathcal{C}$,

$$\mathrm{Isom}(x, x')(f) := \mathrm{Isom}_{\mathcal{F}(Y)}(f^*x, f^*x') = \mathrm{Hom}_{\mathcal{F}(Y)}(f^*x, f^*x')$$

because $\mathcal{F}(Y)$ is a groupoid (hence all arrows are isomorphisms). This depends on choices of pullbacks but we just fix one for all f.

Why is $\mathrm{Isom}(x, x')$ a presheaf?

Say we have have our arrows $Y \xrightarrow{f} X$, and $Z \xrightarrow{g} Y$. Then we get

$$\begin{tikzcd}
(gf)^*x' & g^*f^*x' & f^*x' & x' \\
(gf)^*x & g^*f^*x \arrow[u, "\sim"', "g^*\alpha"] & f^*x \arrow[u, "\sim"', "\alpha"] & x
\end{tikzcd}$$

where $(gf)^*$ and g^*f^* are two choices of pullback. However, note all pullbacks are isomorphic, we see that we get

$$\begin{tikzcd}
(gf)^*x' \arrow[r, "\gamma", "\sim"'] & g^*f^*x' & f^*x' & x' \\
(gf)^*x \arrow[r, "\beta", "\sim"'] & g^*f^*x \arrow[u, "\sim"', "g^*\alpha"] & f^*x \arrow[u, "\sim"', "\alpha"] & x
\end{tikzcd}$$

In particular, this means that β is canonical isomorphism and hence we get (canonical) arrow

$$\begin{gathered}
\mathrm{Isom}(x,x')(f) = \mathrm{Hom}(f^*x, f^*x') \to \mathrm{Isom}(x,x')(gf) = \mathrm{Hom}((gf)^*x, (gf)^*x') \\
\alpha \mapsto \gamma^{-1}(g^*\alpha)\beta
\end{gathered}$$

which concludes $\mathrm{Isom}(x,x')$ is a presheaf (as it is compatible with composition of arrows).

Next, we define fibered products of groupoids. So, unlike normal fibered products in 1-category, now we are working with 2-categories, hence we also need to consider arrows between arrows.

We will start with a diagram of groupoids

$$\begin{tikzcd}
 & \mathcal{G}_1 \arrow[d, "f"] \\
\mathcal{G}_2 \arrow[r, "g"] & \mathcal{G}
\end{tikzcd}$$

We are going to define $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ so that we get the following diagram:

$$\begin{tikzcd}
\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathcal{G}_1 \arrow[d, "f"] \arrow[dl, Rightarrow, "\Sigma"'] \\
\mathcal{G}_2 \arrow[r, "g"] & \mathcal{G}
\end{tikzcd}$$

where Σ is arrow between arrows.

Next, we will define some arbitrary category $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$, then we talk about universal properties that will convince us this is what fibered products should be for groupoids.

Definition 2.2.12

For groupoids $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}$ with $f : \mathcal{G}_1 \to \mathcal{G}$ and $g : \mathcal{G}_2 \to \mathcal{G}$, let $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ be the following category.

The objects are (x, y, σ) where $x \in \mathcal{G}_1, y \in \mathcal{G}_2$ and $f(x) \underset{\sigma}{\xrightarrow{\sim}} g(y)$.

The morphisms for (x',y',σ') to (x,y,σ) will be a pair $(a : x' \to x, b : y' \to y)$ so that we get diagram

$$\begin{tikzcd}
f(x') \arrow[r, "\sim", "f(a)"'] \arrow[d, "\sigma'", "\sim"'] & f(x) \arrow[d, "\sigma", "\sim"'] \\
g(y') \arrow[r, "g(b)", "\sim"'] & g(y)
\end{tikzcd}$$

Next, we need to define p_1 and p_2. They are given by $p_1(x,y,\sigma) = x$ and $p_2(x,y,\sigma) = y$, and $\Sigma(x,y,\sigma) = \sigma$.

To add a few words on Σ, we note $\Sigma : f \circ p_1 \to g \circ p_2$, hence by definition this means we want that, inside $\mathcal{G}_1 \times \mathcal{G}_2$, for any $(x',y',\sigma') \to (x,y,\sigma)$, we get the following commutative square

$$\begin{tikzcd}
f \circ p_1(x',y',\sigma') \arrow[r] \arrow[d, "\Sigma"] & f \circ p_1(x,y,\sigma) \arrow[d, "\Sigma"] \\
g \circ p_2(x',y',\sigma') \arrow[r] & g \circ p_2(x,y,\sigma)
\end{tikzcd}$$

But if you expand the definition of p_i, Σ, this becomes exactly the square in definition of morphisms in $\mathcal{G}_1 \times \mathcal{G}_2$. Hence Σ is indeed natural trans as desired.

Now we talk about the universal property of fiber product. We continue all the notations as in the definition, then for all diagrams

$$\begin{tikzcd}
\mathcal{H} \arrow[rrd, "\alpha"] \arrow[rdd, "\beta"'] & & \\
& & \mathcal{G}_1 \arrow[d, "f"] \arrow[dl, phantom, "\overset{\gamma}{\swarrow}"] \\
& \mathcal{G}_2 \arrow[r, "g"'] & \mathcal{G}
\end{tikzcd}$$

where $\mathcal{H}$ is a groupoid with $\mathcal{H} \xrightarrow{\alpha} \mathcal{G}_1$, $\mathcal{H} \xrightarrow{\beta} \mathcal{G}_2$ and isomorphism (which is natural transformation) $\gamma : f \circ \alpha \to g \circ \beta$, we get unique $(h, \lambda_1, \lambda_2)$ with diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[rrd, bend left, "\alpha"] \arrow[rdd, bend right, "\beta"'] \arrow[rd, dashed, "\exists! h"] \arrow[r, phantom, "\Downarrow \lambda_1"] & & \\
\arrow[r, phantom, "\overset{\lambda_2}{\Rightarrow}"] & \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathcal{G}_1 \arrow[d, "f"] \\
& \mathcal{G}_2 \arrow[r, "g"'] \arrow[ru, phantom, "\overset{\Sigma}{\swarrow}"] & \mathcal{G}
\end{tikzcd}$$

where the two arrows λ_1, λ_2 are natural trans between arrows α (β, respectively) and the composing arrows of h and p_1 (h and p_2, respectively), so that the diagram

$$\begin{tikzcd}
f \circ \alpha \arrow[r, "f(\lambda_1)"] \arrow[d, "\gamma"] & f \circ p_1 \circ h \arrow[d, "\Sigma \circ h"] \\
g \circ \beta \arrow[r, "g(\lambda_2)"] & g \circ p_2 \circ h
\end{tikzcd}$$

commutes. Note here $\Sigma \circ h$ is the same as $\Sigma \circ \mathrm{Id}_h$ where $\mathrm{Id}_h : h \to h$ is the identity natural trans.

Well, why is this object exists? To answer this, we want to construct the unique $h : \mathcal{H} \to \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$. This will be the most "obvious" thing to do, which is $z \mapsto (\alpha(z), \beta(z), \gamma(z))$. We left the details of how h acts on morphisms, as it should be natural.

Now we want to make sure we get the natural transformations λ_1, λ_2. That is, we want a natural trans $\lambda_1 : \alpha \to p_1 \circ h$. This is the same as, for arbitrary z we want to get $\lambda_1(z) : \alpha(z) \to p_1(h(z)) = \alpha(z)$. Well, there is only one natural thing to do, which is take $\lambda_1(z)$ to be identity between $\alpha(z)$ and $\alpha(z)$. We do the same for λ_2.

Next we need to check commutativity of the following diagram

$$\begin{tikzcd}
f \circ \alpha \arrow[r, "f(\lambda_1)"] \arrow[d, "\gamma", "\sim"'] & f \circ p_1 \circ h \arrow[d, "\Sigma \circ h", "\sim"'] \\
g \circ \beta \arrow[r, "g(\lambda_2)"] & g \circ p_2 \circ h
\end{tikzcd}$$

where γ is given by definition:

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathcal{G}_1 \arrow[d, "f"] \arrow[dl, phantom, "\overset{\gamma}{\swarrow}"] \\
\mathcal{G}_2 \arrow[r, "g"'] & \mathcal{G}
\end{tikzcd}$$

Now for any z, we get

$$\begin{tikzcd}
f(\alpha(z)) \arrow[r, "f(\lambda_1(z))"] \arrow[d, "\gamma(z)"] & f(\alpha(z)) \arrow[d, "\Sigma(h(z))"] \\
g(\beta(z)) \arrow[r, "g(\lambda_2(z))"] & g(\beta(z))
\end{tikzcd}$$

but then in particular $\Sigma(h(z)) = \Sigma(\alpha(z), \beta(z), \gamma(z)) = \gamma(z)$ by definition. Hence it is indeed commutative.

Example 2.2.13

Let G be a group acting on a set U via $\sigma : G \times U \to U$. Then we can define a groupoid $[U/G]$ with objects being all elements $x \in U$ and $\mathrm{Hom}(x, x') = \{g \in G : x' = gx\}$. Let $p : U \to [U/G]$ be the projection map, then we have Cartesian diagrams

$$\begin{tikzcd}
G \times U \arrow[r, "\sigma"] \arrow[d, "p_2"] & U \arrow[d, "p"] \\
U \arrow[r, "p"] & {[U/G]}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
G \times U \arrow[r, "(\sigma, p_2)"] \arrow[d] & U \times U \arrow[d, "p \times p"] \\
{[U/G]} \arrow[r, "\Delta"] & {[U/G] \times [U/G]}
\end{tikzcd}$$

To see the first diagram is Cartesian, observe that, for any groupoids the fibered product is defined by (x, y, σ) with $x \in \mathcal{G}_1, y \in \mathcal{G}_2$ and $\sigma : f(x) \xrightarrow{\sim} g(y)$, where the notations are from the above definition. For us, this means we require $x, y \in U$ together with an isomorphism $[x] \cong [y] \in [U/G]$. However,

$[x] \cong [y] \in [U/G]$ if and only if $x = gy$ for some $g \in G$. This shows the triple (x, y, σ) in the definition of fibered product of groupoids can be identified by a pair (x, g), where the identification is given by $(x, g) \mapsto (x, gx, g)$, and hence the two projections are precisely σ and p_2. That is, the first diagram is indeed Cartesian.

For the second diagram, take $[x] \in [U/G]$ and $(y, z) \in U \times U$, we must also have an isomorphism $([x], [x]) \cong ([y], [z])$, i.e. we need $x = g_1 y$ and $x = g_2 z$ for some $g_1, g_2 \in G$. This is the same as $(g_2^{-1} g_1, y)$, where we can identify $[x] \in [U/G]$ by $G \times U \to [U/G]$, and (y, z) under the map (σ, p_2).

This concludes the definition of fibered products of groupoids, and we are heading to define fibered products of categories fibered in groupoids.

For this, let $\mathcal{F}_i \to \mathcal{C}$ be categories fibered in groupoids. Then we want to have a diagram

$$\begin{tikzcd} \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2 \arrow[r] \arrow[d] & \mathcal{F}_1 \arrow[d, "f"] \\ \mathcal{F}_2 \arrow[r, "g"] & \mathcal{F} \end{tikzcd}$$

so that for all $\mathcal{H} \to \mathcal{F}_1$ and $\mathcal{H} \to \mathcal{F}_2$ we get unique arrow $\mathcal{H} \to \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$ with additional arrows between the arrows.

We want $\mathcal{G} = \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$ to have the property that

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_1) \times_{\mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F})} \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_2)$$

However, on the RHS, they are just fibered products of groupoids, and by 2-Yoneda lemma, this determines $\mathcal{G}$ if the RHS exists.

Proposition 2.2.14: Olsson, Prop 3.4.13

Let $\mathcal{F}_1, \mathcal{F}_2 \to \mathcal{C}$. Then the fibered product $\mathcal{G} = \mathcal{F}_1 \times_{\mathcal{C}} \mathcal{F}_2$ exists.

Explicitly, the fiebred product is described as follows. Suppose we have

$$\begin{tikzcd} & \mathcal{F}_1 \arrow[d, "f"] \\ \mathcal{F}_2 \arrow[r, "g"] & \mathcal{F} \end{tikzcd}$$

a diagram of categories fibered in groupoids over $\mathcal{C}$, with projections $p_{\mathcal{F}_1}, p_{\mathcal{F}_2}$ and $p_{\mathcal{F}}$. Then, we define $\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$ as the category of triples (x, y, γ), with $x \in \mathcal{F}_1$, $y \in \mathcal{F}_2$ such that $p_{\mathcal{F}_1}(x) = p_{\mathcal{F}_2}(y) =: S$, and $\gamma : f(x) \xrightarrow{\sim} g(y)$ is an isomorphism in the category $\mathcal{F}(S)$. A morphism $(x_1, y_1, \gamma_1) \to (x_2, y_2, \gamma_2)$ is given by $(h, a : x_1 \xrightarrow{\sim} x_2, b : y_1 \xrightarrow{\sim} y_2)$ where $h : p_{\mathcal{F}_1}(x_1) = p_{\mathcal{F}_2}(y_1) \to p_{\mathcal{F}_2}(y_2) = p_{\mathcal{F}_1}(x_2)$ is a morphism in $\mathcal{C}$, and a, b are morphisms over h (i.e. $p_{\mathcal{F}_1}(a) = h = p_{\mathcal{F}_2}(b)$), such that

$$\begin{tikzcd} f(x_1) \arrow[r, "f(a)"] \arrow[d, "\gamma_1"] & f(x_2) \arrow[d, "\gamma_2"] \\ g(y_1) \arrow[r, "g(b)"] & g(y_2) \end{tikzcd}$$

Then, let p_1, p_2 be the two projections $(x, y, \gamma) \mapsto x$ and $(x, y, \gamma) \mapsto y$. Then we can define natural transformation $\alpha : f \circ p_1 \xrightarrow{\sim} g \circ p_2$ by $\alpha_{(x,y,\gamma)} : f(x) \xrightarrow{\gamma} g(y)$.

In particular, to give an arrow $\mathcal{T} \to \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$ is the same as to give $q_1 : \mathcal{T} \to \mathcal{F}_1$, $q_2 : \mathcal{T} \to \mathcal{F}_2$, and natural transformation $\tau : f \circ q_1 \xrightarrow{\sim} g \circ q_2$, that makes the appropriate diagram commutes (figure out what this diagram should be).

Proposition 2.2.15

Let $p : \mathcal{F} \to \mathcal{C}$ be a category fibered in groupoids, $x, x' \in \mathcal{F}(X)$ with $X \in \mathcal{C}$. Then there is a Cartesian diagram

$$\begin{tikzcd}
\operatorname{Isom}(x, x') \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "{(x,x')}"] \\
\mathcal{F} \arrow[r, "\Delta"] & \mathcal{F} \times_{\mathcal{C}} \mathcal{F}
\end{tikzcd}$$

Proof. An element of the fibered product $X \times_{\mathcal{F} \times_{\mathcal{C}} \mathcal{F}} \mathcal{F}$ is a triple (h_1, h_2, γ) where $h_1 : Y \to X \in \mathcal{C}/X$ (recall Example 2.2.3), $h_2 \in \mathcal{F}$, so that $p_X(h_1) = p_{\mathcal{F}}(h_2)$. However, $p_X(h_1) = Y$ and hence we must have $p_{\mathcal{F}}(h_2) = Y$, i.e. $h_2 \in \mathcal{F}(Y)$. Next, we require an isomorphism $\gamma : (x, x')(h_1) \xrightarrow{\sim} \Delta(h_2)$ in the category $\mathcal{F}(Y) \times \mathcal{F}(Y)$. Now note $(x, x')(h_1)$ is simply $(h_1^* x, h_1^* x')$, while $\Delta(h_2) = (h_2, h_2)$. Hence, we just get isomorphism $\gamma : (h_1^* x, h_1^* x') \xrightarrow{\sim} (h_2, h_2)$. However, isomorphisms in the category $\mathcal{F}(Y) \times \mathcal{F}(Y)$ means $h_1^* x \cong h_2$ and $h_1^* x' \cong h_2$ in $\mathcal{F}(Y)$, i.e. γ is the same as an isomorphism $h_1^* x \cong h_1^* x'$ in $\mathcal{F}(Y)$. In other word, we have identified $X \times_{\mathcal{F} \times_{\mathcal{C}} \mathcal{F}} \mathcal{F}$ with $h_1 : Y \to X$ in $\mathcal{C}$, together with an isomorphism $h_1^* x \to h_1^* x'$, i.e. this is indeed just $\operatorname{Isom}(x, x')(h_1)$, and hence we are done.

Chapter 3

Categorical Stacks

白衣苍狗变浮云，千古功名一聚尘。好是悲歌将进酒，不妨同赋惜馀春。
风光全似中原日，臭味要须我辈人。雨后飞花知底数？醉来赢取自由身。

张元干

3.1 (Cat) Stack

This concludes the topic about fibered products, and we are back to descents. After this, we will define what stacks are.

The idea of descents should be that, they are like sheaf axiom for fibered categories.

Example 3.1.1

Let X be a scheme and C be the category $\mathrm{Op}(X)$. Then consider $\mathcal{F} = (\mathbf{Vect}) \to \mathcal{C}$ where $\mathcal{F}(U)$ be the category of vector bundles on U. Then, if $U = \bigcup_i U_i$, a vector bundle on U is not equivalent to $\mathcal{E}_i$ on U_i with double intersections isomorphic (i.e. $\sigma_{ij} : \mathcal{E}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{E}_j|_{U_{ij}}$). Indeed, just recall the example in the introduction of chapter 1.

In this case, the naive sheaf axioms fit into the picture, i.e. we get

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij})$$

and this diagram is not exact.

We are missing the "cocycle" condition (to make the above diagram exact/equalizer). This means that, on $U_{ijk} = U_i \cap U_j \cap U_k$, we get diagram

$$\begin{tikzcd} \mathcal{E}_i|_{U_{ijk}} \arrow[r, "\sigma_{ij}"] \arrow[dr, "\sigma_{ik}"'] & \mathcal{E}_j|_{U_{ijk}} \arrow[d, "\sigma_{jk}"] \\ & \mathcal{E}_k|_{U_{ijk}} \end{tikzcd}$$

In other word, the right "exact" diagram we need will be something like

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij}) \Rrightarrow \prod_{ijk} \mathcal{F}(U_{ijk})$$

Therefore, we want to formalize this triple arrow thing in fibered categories.

Let $p : \mathcal{F} \to \mathcal{C}$ be a fibered category. Given $f : X \to Y$ in $\mathcal{C}$, let $\mathcal{F}(X \xrightarrow{f} Y)$ be the category defined as follows (this is called category of descent data). The object should be (E, σ) with $E \in \mathcal{F}(X)$ and

$$X \times_Y X \times_Y X \Rrightarrow X \times_Y X \rightrightarrows X \xrightarrow{f} Y$$

where the triple arrows are p_{12}, p_{13}, p_{23} and the double arrows are p_1, p_2, and $\sigma : p_1^* E \to p_2^* E$ is an isomorphism in $\mathcal{F}(X \times_Y X)$ such that we get the following commutative diagram

$$\begin{tikzcd} p_{13}^* p_1^* E \arrow[r, "="] \arrow[d, "p_{13}^*\sigma"] & p_{12}^* p_1^* E \arrow[r, "p_{12}^*\sigma"] & p_{12}^* p_2^* E \arrow[d, "="] \\ p_{13}^* p_2^* E \arrow[r, "="] & p_{23}^* p_2^* E & p_{23}^* p_1^* E \arrow[l, "p_{23}^*\sigma"] \end{tikzcd} \tag{Eq. 3.1.1}$$

where $=$ means canonical isomorphism. This is called the cocycle condition.

Remark 3.1.2

Here is just a brife recall of what all the above notations (i.e. $p_i^* E$, p_{ij}, etc) means.

First, recall that $p_i^* E$ are defined as the pullback of the following diagram

$$\begin{tikzcd} p_i^* E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \times_Y X \arrow[r, "p_i"] & X \end{tikzcd}$$

Similarly, $p_{jk}^* p_i E^*$ are defined as the pullback of the following diagram

$$\begin{tikzcd} p_{jk}^* p_i^* E \arrow[r] \arrow[d] & p_i^* E \arrow[d] \\ X \times_Y X \times_Y X \arrow[r, "p_{jk}"] & X \times_Y X \end{tikzcd}$$

Second, we note p_{ij} are projections come from the universal property of (fibered) products. In other word, note we would define $X_1 \times_S X_2 \times_S X_3$ as the unique object satisfies the following diagram

$$\begin{tikzcd}
& X_1 \times X_3 \arrow[dl] \arrow[dd, dashed] & & X_1 \times_Y X_2 \times_Y X_3 \arrow[ll] \arrow[dl] \arrow[dd] \\
X_1 \arrow[dd] & & X_1 \times X_2 \arrow[ll] \arrow[dd] & \\
& X_3 \arrow[dl, dashed] & & X_2 \times X_3 \arrow[ll, dashed] \arrow[dl] \\
Y & & X_2 \arrow[ll] &
\end{tikzcd}$$

Finally, a word on the isomorphisms $p_{jk}^* p_i^* E$. Continue with the above diagram (where now we let $X_1 = X_2 = X_3 = X$), we get the following

$$\begin{tikzcd}
& p_1^* E \arrow[dl] \arrow[dd, dashed] & & p_{13}^* p_1^* E \cong p_{12}^* p_1^* E \arrow[ll] \arrow[dl] \arrow[dd] \\
E \arrow[dd] & & p_1^* E \arrow[ll] \arrow[dd] & \\
& X_1 \times X_3 \arrow[dl, "p_1"'] \arrow[dd, dashed] & & X_1 \times_Y X_2 \times_Y X_3 \arrow[ll, "p_{13}"'] \arrow[dl, "p_{12}"'] \arrow[dd] \\
X_1 \arrow[dd] & & X_1 \times X_2 \arrow[ll, "p_1"'] \arrow[dd] & \\
& X_3 \arrow[dl, dashed] & & X_2 \times X_3 \arrow[ll, dashed] \arrow[dl] \\
Y & & X_2 \arrow[ll] &
\end{tikzcd}$$

where the two pullbacks along p_{13} and p_{12} must be the same object living over $X_1 \times X_2 \times X_3$, hence the canonical isomorphisms between $p_{13}^* p_1^* E \cong p_{12}^* p_1^* E$. The others are similar.

We are finally going to define what a cat stack is:

It is a cat that stacks on your computer such that no matter how you pull them back they are still cat stacks on your computer!

In the above, let $p : \mathcal{F} \to \mathcal{C}$ be a fibered category (not necessarily fibered in groupoids). Given $X \xrightarrow{f} Y$ in $\mathcal{C}$, we defined a category $\mathcal{F}(X \xrightarrow{f} Y)$ be the category of descent data.

The point is that, we get $\mathcal{F}(Y) \xrightarrow{\epsilon} \mathcal{F}(X \xrightarrow{f} Y)$ by

$$F \mapsto (f^*F, \sigma_{\text{can}})$$

This is because, $p_1^*f^*F$ and $p_2^*f^*F$ are pullbacks of F along $g : X \times_Y X \to Y$. For example, we get $p_1^*f^*F$ by the following diagram (where g is equal both fp_1 and fp_2 at the same time)

$$\begin{tikzcd}
p_1^*f^*F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & f^*F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[d, "p"] \\
X \times_Y X \arrow[r, "p_1"] \arrow[rr, bend right, "g=fp_1=fp_2"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Therefore we get a canonical map $\sigma_{\text{can}} : p_1^*f^*F \xrightarrow{\sim} p_2^*f^*F$.

In this case, we say f is an ***effective descent morphism*** if ϵ is equivalence of categories. In this case we also say $\mathcal{F}$ satisfies descent for f.

Before we say explain what this means, we recall the morphisms of $\mathcal{F}(X \xrightarrow{f} Y)$ is the following. From (E', σ') to (E, σ), a morphism is $\alpha : E' \to E$ in $\mathcal{F}(X)$ such that we get the following commuting diagram

$$\begin{tikzcd}
p_1^*E' \arrow[r, "p_1^*\alpha"] \arrow[d, "\sigma'"] & p_1^*E \arrow[d, "\sigma"] \\
p_2^*E' \arrow[r, "p_2^*\alpha"] & p_2^*E
\end{tikzcd}$$

More generally, we can define $\mathcal{F}(\{X_i \xrightarrow{f_i} Y\}_{i \in I})$ as $(\{E_i\}_{i\in I}, \{\sigma_{ij}\}_{i,j})$ such that $E_i \in \mathcal{F}(X_i)$ and $\sigma_{ij} : p_i^*E_i \xrightarrow{\sim} p_j^*E_j$ is an isomorphism, where

$$\begin{tikzcd}
X_{ij} := X_i \times_Y X_j \arrow[r, "p_j"] \arrow[d, "p_i"] \arrow[dr, phantom, "\square"] & X_j \arrow[d, "f_i"] \\
X_i \arrow[r, "f_i"] & Y
\end{tikzcd}$$

We also need the cocycle condition $\sigma_{jk} \circ \sigma_{ij} = \sigma_{ik}$.

Normally, we do not need to think about this more general case because of the following lemma.

Lemma 3.1.3: Olsson, Lemma 4.2.7

Assume coproducts exist in $\mathcal{C}$ and coproducts commute with fiber products when they exists. Assume for all sets of objects $\{X_i\}_{i\in I}$ in $\mathcal{C}$ the natural map $\mathcal{F}(\coprod X_i) \to \prod \mathcal{F}(X_i)$ is an equivalence. Then if $\{X_i \to Y\}$ are morphisms in $\mathcal{C}$ and $Q = \coprod_i X_i$, then $\mathcal{F}(Y) \to \mathcal{F}(\{X_i \to Y\})$ is equivalence (of categories) if and only if $\mathcal{F}(Y) \to \mathcal{F}(Q \to Y)$ is equivalence (of categories).

Note this is coproduct, not co-fiber product. Co-fiber product may not exists in schemes. On the other hand, coproducts exists in schemes, as they are just disjoint union of schemes.

Thus, the point is that, we can always think $\mathcal{F}(\{X_i \to Y\})$ as $\mathcal{F}(\coprod X_i \to Y)$, which means we back to the first case.

Before we mentioned informally what stacks are. This time we write down the formal definition.

Definition 3.1.4

A ***stack on a site*** $\mathcal{C}$ is a category fibered in groupoids $p : \mathcal{F} \to \mathcal{C}$, such that descent data is effective for covering maps, i.e. if $\{X_i \to Y\}_i \in \mathrm{Cov}(Y)$, then $\mathcal{F}(Y) \xrightarrow{\epsilon} \mathcal{F}(\{X_i \to Y\})$ is an equivalence between categories.

In short, stacks are just category fibered in groupoids where descent holds.

Remark 3.1.5

This is what we really call categorical stack, as we haven't done any actual geometry yet. Fibered categories are analogous to presheaves (they are presheaves when fibered in sets). So stacks are presheaves where sheaf axiom holds.

What this means is that, for example, take fppf topology on schemes and choose any sheaf $\mathcal{F}$. Then we say $\mathcal{F}$ is "geometric" if $\mathcal{F} = h_X$ for some scheme X. Those $\mathcal{F}$ are of course example of stacks.

Hence, in general, we want "representable stacks" (the technical term is algebraic stacks, or Artin stack) instead of arbitrary stacks.

Alternatively, we can define stacks as follows. Suppose $p : \mathcal{F} \to \mathcal{C}$ be category fibered in groupoids, then it is a stack if, for all covers $\{Y_i \to Y\}_{i\in I}$ we have:

1. (Morphism Glue): For objects $E, E' \in \mathcal{F}(Y)$ and morphisms $\sigma_i : E|_{Y_i} \to E'$ such that $\sigma_i|_{Y_{ij}} \cong \sigma_j|_{Y_{ij}}$, there is unique $\sigma : E \to E'$ so $\sigma|_{Y_i} = \sigma_i$.
2. (Objects Glue): For objects E_i over Y_i and isomorphisms $\sigma_{ij} : E_i|_{Y_{ij}} \to E_j|_{Y_{ij}}$, if we have $\sigma_{jk}|_{Y_{ijk}} \circ \sigma_{ij}|_{Y_{ijk}} = \sigma_{ik}|_{Y_{ijk}}$ on Y_{ijk}, then there exists E over Y and isomorphisms $\sigma_i : E|_{S_i} \to E_i$ so $\sigma_{ij} \circ \sigma_i|_{S_{ij}} = \sigma_j|_{S_{ij}}$ on S_{ij}.

In the above, for any finite index set $J = \{j_1, \ldots, j_k\} \subseteq I$ we define $Y_J := Y_{j_1} \times_Y \ldots \times_Y Y_{j_k}$.

We can also describe the above condition as simply the exactness of the following sequence

$$\mathcal{F}(Y) \longrightarrow \prod_i \mathcal{F}(Y_i) \rightrightarrows \prod_{ij} \mathcal{F}(Y_{ij}) \Rrightarrow \prod_{ijk} \mathcal{F}(Y_{ijk})$$

The equivalence of those conditions are left as an exercise (hint: equivalence of categories if and only if fully faithful and essentially surjective, and then the object glues tells us the map is essentially surjective, while the first condition says fully faithful).

Our next goal is to get a feeling about cat stacks, and then we try to find what would be a nice notion for algebraic stacks.

Well, I lied. Here is the punch line for what algebraic stacks are.

Remark 3.1.6: Spoiler Alert

The idea for algebraic stacks is that, if $\mathcal{F}$ is a stack over schemes. To import/involve geometry, we require that there exists X and arrow $X = h_X \to \mathcal{F}$, so that $h_X \twoheadrightarrow \mathcal{F}$ is a "smooth cover". This makes no sense, as we don't know what smooth covers are.

Thus, what should smooth cover $X \twoheadrightarrow \mathcal{F}$ be? Well, it should have the property that, for any scheme T, if we take fibered product

$$\begin{tikzcd} X \times_{\mathcal{F}} T \arrow[r] \arrow[d, two heads, "g"', "sm"] \arrow[dr, phantom, "\square"] & X \arrow[d, two heads] \\ T \arrow[r] & \mathcal{F} \end{tikzcd}$$

then the arrow $X \times_{\mathcal{F}} T \to T$ is a smooth cover. Well, this helps a little bit, as now our base becomes a scheme T. But, what is $X \times_{\mathcal{F}} T$? We don't know, hence we just insists that it should be nice, i.e. it should be a scheme by definition (this is actually not the full definition, i.e. the actual def is $X \times_{\mathcal{F}} T$ should be algebraic space).

In other word, algebraic stacks are stacks over scheme that we get a smooth cover $X \twoheadrightarrow \mathcal{F}$, where smooth cover means when we pullback along scheme T we always get $X \times_{\mathcal{F}} T$ be a scheme and $X \times_{\mathcal{F}} T \twoheadrightarrow T$ is a smooth cover of schemes.

Next, we consider an example of effective descent morphism.

Proposition 3.1.7

If $f : X \to Y$ admits a section $s : Y \to X$, i.e. $fs = \mathrm{Id}_Y$, then descent data is effective for f.

Note this holds for any category.

Proof. We have $\mathcal{F}(Y) \xrightarrow{\epsilon} \mathcal{F}(X \xrightarrow{f} Y)$ by $F \mapsto (f^*F, \sigma_{\mathrm{can}})$. Thus we define $\eta : \mathcal{F}(X \to Y) \to \mathcal{F}(Y)$ by

$$(E, \sigma) \mapsto s^*E$$

where we recall $E \in \mathcal{F}(X)$ and $\sigma : p_1^*E \xrightarrow{\sim} p_2^*E$. Let's check this is what we wanted. Indeed, we see $\eta\epsilon(F) = \eta(f^*F, \sigma_{\text{can}}) = s^*f^*F \cong (fs)^*F \cong F$.

Now we need to go the other way, i.e. we start with (E,σ). We get the diagram

$$\begin{tikzcd}
X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Then, we get

$$\begin{tikzcd}
X \arrow[dr, dotted, "h"] \arrow[drr, bend left, "\mathrm{Id}"] \arrow[ddr, bend right, "sf"'] & & \\
& X \times_Y X \arrow[r, "p_1"'] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

This commutes because $f \circ \mathrm{Id} = f$ and $fsf = f$. Hence the dotted arrow h in the above exists.

By $\sigma : p_1^*E \xrightarrow{\sim} p_2^*E$, we get diagram

$$\begin{tikzcd}
h^*p_1^*E \arrow[r, "\sim", "h^*\sigma"'] \arrow[d, "="] & h^*p_2^*E \arrow[d, "="] \\
E & f^*s^*E = \epsilon\eta(E,\sigma)
\end{tikzcd}$$

Here $h^*p_2^* \cong (p_2h)^* \cong (sf)^* \cong f^*s^*$ hence $h^*p_2^*E \cong f^*s^*E$ and similarly $h^*p_1^*E \cong E = \mathrm{Id}^*E$. This gives an arrow $E \to f^*s^*E$ which we denote by $\rho_{(E,\sigma)}$. Next, we need to show this map is compatible with σ, i.e. we want the following diagram

$$\begin{tikzcd}
p_1^*E \arrow[r, "\sigma"] \arrow[d, "p_1^*\rho_{(E,\sigma)}"] & p_2^*E \arrow[d, "p_2^*\rho_{(E,\sigma)}"] \\
p_1^*f^*s^*E \arrow[r, "\sigma_{\text{can}}"] & p_2^*f^*s^*E
\end{tikzcd}$$

But then by the above we see[1] $p_i^*\rho_{(E,\sigma)} = p_i^*(\mathrm{Id}, sf)^*\sigma$ which is equal $(p_i, sfp_i)^*\sigma$ where we recall $fp_1 = fp_2$ and thus $p_i^*(\mathrm{Id}, sf)^*\sigma \cong (\mathrm{Id} \circ p_i, sfp_i)^*\sigma$. But then we see the cocycle condition says $(p_1, sfp_1)^*\sigma = (p_2, sfp_2)^*\sigma \circ (p_1, p_2)^*\sigma$, i.e.

$$p_1^*\rho_{(E,\sigma)} = p_2^*\rho_{(E,\sigma)} \circ \sigma$$

Alternatively, this argument above is the same as pullback the cocycle diagram Eq. 3.1.1 using $(\mathrm{Id}_{X\times X}, sfp_1)$.

Hence, we see this construction yields an isomorphism $(E,\sigma) \xrightarrow{\sim} \epsilon\eta(E,\sigma)$ as desired.

[1] To see this, just recall $h = (\mathrm{Id}, sf)$ and $\rho_{(E,\sigma)}$ is defined by $h^*\sigma$

This is a silly example, but it is actually very useful, as we can frequently reduce to the case where we have sections.

Now we consider descent for sheaves. Let $\mathcal{C}$ be a site where finite limits exist. Then take $f : X \to Y$ in $\mathcal{C}$, we get $\widetilde{\mathcal{C}/X} \underset{f_*}{\overset{f^*}{\longleftrightarrow}} \widetilde{\mathcal{C}/Y}$ maps between their topoi (here we use $\widetilde{\mathcal{C}/X}$ to denote topos). Here we get $(f^*\mathcal{F})(W \to X) = \mathcal{F}(W \to X \xrightarrow{f} Y)$. So, $f^*g^* = (gf)^*$ are equivalent.

Let $p : (\mathbf{Sh}) \to \mathcal{C}$ be the following category. The objects are (X, E) where $X \in \mathcal{C}$ and $E \in \widetilde{\mathcal{C}/X}$. The morphisms from (X, E) to (Y, F) are given by $X \xrightarrow{f} Y$ plus $E \to f^*F$. The projection is $p(X, E) = X$. We note this is a fibered category that's not fibered in groupoids.

Theorem 3.1.8

If $f : X \to Y$ is a covering in $\mathcal{C}$, then f is effective descent morphism for $(\mathbf{Sh})$.

Proof. Consider $(\mathbf{Sh})(X \to Y)$. This has objects (E, σ) where $E \in \widetilde{\mathcal{C}/X}$ and $\sigma : p_1^*E \xrightarrow{\sim} p_2^*E$ in $\widetilde{\mathcal{C}/(X \times_Y X)}$ that satisfies cocycle condition. For

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[rr, bend right, "g"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

we want to construct an inverse functor $\eta : (\mathbf{Sh})(X \to Y) \to (\mathbf{Sh})(Y)$.

In this case, we get diagram

$$\begin{tikzcd} g_* p_2^* E \arrow[r, "g_*\sigma^{-1}", "\sim"'] \arrow[d, "="] & g_* p_1^* E \arrow[d, "="] \\ f_*(p_2)_* p_2^* E & f_*(p_1)_* p_1^* E \\ f_* E \arrow[u, "f_*(\text{adjunction})"] & f_* E \arrow[u] \end{tikzcd}$$

This is not a commutative diagram, thus we want to take equalizer. That is, we want to take $\mathrm{Eq}(f_*E \underset{\sigma^{-1}p_2^*}{\overset{p_1^*}{\rightrightarrows}} g_*p_1^*E) =: \eta((E, \sigma))$ as our definition. Here we abused notations. In particular, we write p_1^* to mean the arrow $f_*E \to f_*(p_1)_*p_1^*E$ given by apply f_* to the adjunction map $E \to (p_1)_*p_1^*E$ then take the reverse arrow of the arrow $g_*p_1^*E \to f_*(p_1)_*p_1^*E$. Similarly p_2^* is the arrow $f_*E \to g_*p_2^*E$ obtained from the above diagram. Also, that $\sigma^{-1}p_2^*$ is also abuse of notations, as what we really meant is $(g_*\sigma^{-1})p_2^*$ as in the above diagram.

Then, we claim $\mathrm{Id} \xrightarrow{\sim} \eta \circ \epsilon$. Indeed, if $F \in (\mathbf{Sh})(Y)$, then $\epsilon(F) = (f^*F, \sigma_{\mathrm{can}})$. Then $\eta\epsilon(F) = \eta(f^*F, \sigma_{\mathrm{can}}) = \mathrm{Eq}(f_*f^*F \underset{\sigma_{\mathrm{can}}^{-1}p_2^*}{\overset{p_1^*}{\rightrightarrows}} g_*p_1^*f^*F)$. Now note $p_1^*f^*F = g^*F$ and hence we just want to show that F is an equalizer of the arrow $f_*f^*F \rightrightarrows g_*g^*F$, then it will conclude $\eta\epsilon(F) = F$, where we note there is a natural map $F \to f_* f^* F$, i.e. we want to show $F \to f_* f^* F \rightrightarrows g_* g^* F$ is an equalizer diagram.

To do this, we prove it on Z-valued points, i.e. we take arbitrary $Z \to Y$, and we show it holds when we apply F to Z. Let

$$\begin{tikzcd}
Z \arrow[r] & Y \\
X_Z \arrow[u, two heads, "\text{covering}"] \arrow[r] \arrow[ur, phantom, "\square"] & X \arrow[u, two heads, "f"'] \\
X_Z \times_Z X_Z \arrow[u, shift left] \arrow[u, shift right] \arrow[r] \arrow[ur, phantom, "\square"] & X \times_Y X \arrow[u, shift left] \arrow[u, shift right]
\end{tikzcd}$$

Then, we see we get

$$F(Z) \longrightarrow F(X_Z) = f_* f^* F(Z \to Y) \rightrightarrows F(X_Z \times_Z X_Z) = g_* g^* F(Z \to Y)$$

That is exactly by def of sheaf an equalizer. Hence we see $\eta \circ \epsilon \cong \mathrm{Id}$ by Yoneda.

Next, we let (E, σ) be given and set $F := \eta(E, \sigma)$. Then $F \to f_* E$ by construction and so we get $f^* F \to f^* f_* E \to E$, i.e. we get canonical map $\rho(E, \sigma) : (f^* F, \sigma_{\text{can}}) = \epsilon\eta((E, \sigma)) \to (E, \sigma)$. We want to show $\rho(E, \sigma)$ is isomorphism.

To show $\rho(E, \sigma)$ is isomorphism, it is okay to do that locally on Y (left as exercise). To say do this locally, we mean that if

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
Y' \arrow[r, two heads, "g"] & Y
\end{tikzcd}$$

then g^* on topoi is exact, so $g^* \circ \text{equalizer} = \text{equalizer} \circ g^*$. In other word, we get commutative diagram

$$\begin{tikzcd}
(\mathbf{Sh})(X' \to Y') \arrow[d, "\eta'"] & (\mathbf{Sh})(X \to Y) \arrow[l, "(g')^*"'] \arrow[d, "\eta"] \\
(\mathbf{Sh})(Y') & (\mathbf{Sh})(Y) \arrow[l, "g^*"]
\end{tikzcd}$$

Then $(g')^* \rho(E, \sigma) = \rho((g')^* E, (g')^* \sigma)$. In particular, we see ρ' (this is the image of $\rho(E, \sigma)$ in $(\mathbf{Sh})(X' \to Y')$ in the above diagram) is isomorphism implies $\rho(E, \sigma)$ is isomorphism (this claim is left as exercise).

Now here is the trick: since we can take any cover g, we choose $g = f$. Now we get

$$\begin{tikzcd}
X \times_Y X \arrow[r] \arrow[d, "f'"] & X \arrow[d] \\
X \arrow[r, two heads, "f"] \arrow[u, bend left, "s'"] \arrow[ur, phantom, "\square"] & Y
\end{tikzcd}$$

and s' is a section via the diagonal map, i.e. $x \mapsto (x, x) \in X \times_Y X$. Hence by the last theorem, we are done.

As a corollary of the theorem we proved above, we get

Proposition 3.1.9

Let X, Y, S be schemes with diagram:

$$\begin{tikzcd}
Y'' \arrow[r, shift left] \arrow[r, shift right] \arrow[dd, bend right] & Y' \arrow[r] \arrow[dd, bend left] & Y \arrow[dd, bend left] \\
X'' \arrow[u] \arrow[d] \arrow[r, shift left] \arrow[r, shift right] & X' \arrow[u, "f'"] \arrow[d] \arrow[r] & X \arrow[d] \\
S'' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & S' \arrow[r, two heads, "fppf", "g"'] & S \\
S' \times_S S' & &
\end{tikzcd}$$

where $X' = X \times_S S'$, $X'' = X \times_S S''$, and similarly for Y', Y''. THe $f' : X' \to Y'$ over S' is such that $p_1^* f' = p_2^* f'$. Then there exists unique $f : X \to Y$ over S so that $g^* f = f'$.

Proof. We showed a big theorem: h_X, h_Y are fppf sheaves. Thus f' yields $h_{X'} \to h_{Y'}$. In particular, $p_1^* f' = p_2^* f'$ means this extends to $(h_{X'}, \sigma_{\text{can}}) \to (h_{Y'}, \sigma_{\text{can}})$ in $(\mathbf{Sh})(S' \twoheadrightarrow S)$. By big theorem last time, we see $(\mathbf{Sh})(S) \xrightarrow{\sim} (\mathbf{Sh})(S' \twoheadrightarrow S)$ and hence this arrow $(h_{X'}, \sigma_{\text{can}}) \to (h_{Y'}, \sigma_{\text{can}})$ correspond to an arrow $h_X \to h_Y$. Now Yoneda lemma tells us we have the desired arrow $f : X \to Y$.

Next, we talk about variant of descent for sheaves. Let $\mathscr{O}$ be a sheaf of rings on a site $\mathcal{C}$. For all $X \in \mathcal{C}$, let $\mathscr{O}_X \in \widetilde{\mathcal{C}/X}$ be defined by $\mathscr{O}_X(Y \to X) := \mathscr{O}(Y)$. Then for all $f : X \to Y$, we get a map $(\widetilde{\mathcal{C}/X}, \mathscr{O}_X) \to (\widetilde{\mathcal{C}/Y}, \mathscr{O}_Y)$ map of "ringed topoi", i.e. map of topoi plus $\mathscr{O}_Y \to f_* \mathscr{O}_X$.

We will show this is almost a stack.

For $X \in \mathcal{C}$, let Mod_X be the category of $\mathscr{O}_X$-modules in $\widetilde{\mathcal{C}/X}$. Then for all $f : Y \to X$, we get $f^* : \text{Mod}_X \to \text{Mod}_Y$ by $(f^* M)(Z \to Y) := M(Z \to Y \to X)$.

Now we define fibered category $\text{MOD} \xrightarrow{p} \mathcal{C}$ as follows: it has object (E, X) where $X \in \mathcal{C}$, $E \in \text{Mod}_X$. The morphisms are $(F, Y) \to (E, X)$ is $f : Y \to X$ in $\mathcal{C}$ and $\epsilon : F \to f^* E$ in Mod_Y.

Theorem 3.1.10

For all $f : Y \to X$ covers in $\mathcal{C}$, $\text{Mod}_X \xrightarrow{\sim} \text{MOD}(Y \to X)$.

Now we have defined modules, the next topic is of course quasi-coherent sheaves.

Let S be a scheme, $\mathcal{C} = ((\mathbf{Sch})/S)_{\text{fppf}}$ be the fppf site associated with $(\mathbf{Sch})/S$.

Now $\mathcal{O}$ be the presheaf of rings on $\mathcal{C}$ defined as: $\mathcal{O}(T \to S) := \Gamma(\mathcal{O}_T) = \mathrm{Hom}_S(T, \mathbb{A}^1_S)$. This is just the global sections, i.e. it is the sheaf represented by $\mathbb{A}^1_S$. In other word, $\mathcal{O} = h_{\mathbb{A}^1_S}$ and hence we see this is a fppf sheaf as $h_{\mathbb{A}^1_S}$ is fppf sheaf.

Next, we want to figure out what's a reasonable notion of quasi-coherent for fppf topology.

For any scheme S, let $(\mathbf{Qcoh})(S)$ be the category of quasi-coherent $\mathcal{O}_S$-modules in Zariski topology, i.e. for S_{Zar}.

Given $\mathscr{F} \in (\mathbf{Qcoh})(S)$ we get $\mathscr{F}_{big}$, a presheaf of $\mathcal{O}$-modules on $\mathcal{C} = ((\mathbf{Sch})/S)_{\text{fppf}}$, defined as follows

$$\mathscr{F}_{big}(T \xrightarrow{f} S) := (f^*\mathscr{F})(T)$$

Note this depends on choice of pullback.

Lemma 3.1.11

$\mathscr{F}_{big}$ is an fppf sheaf.

Proof. Recall from awhile ago, to prove $\mathscr{F}_{big}$ is an fppf sheaf, we just need to check:

1. $\forall T \to S$, $\mathscr{F}_{big}|_{T_{Zar}}$ is a sheaf, and
2. sheaf condition on fppf arrow $\operatorname{Spec} B \twoheadrightarrow \operatorname{Spec} A$.

To see (1), we see it is enough to show $\mathscr{F}_{big}|_{T_{Zar}}$ is sheaf for small Zariski site. However, we see this is clearly a sheaf, because

$$\mathscr{F}_{big}|_{T_{Zar}} = f^*\mathscr{F} \in (\mathbf{Qcoh})(T)$$

by definition. So it is indeed a sheaf.

For (2), let $f : \operatorname{Spec} A \to S$ and $f^*\mathscr{F}$ be M an A-module. We need to check

$$\mathscr{F}_{big}(A) = M \longrightarrow \mathscr{F}_{big}(B) = M_B \rightrightarrows \mathscr{F}_{big}(B \otimes_A B) = M_{B \otimes_A B}$$

We showed this when showing schemes are fppf sheaves.

So, $\mathscr{F} \in (\mathbf{Qcoh})(S)$ yields $\mathscr{F}_{big} \in \widetilde{((\mathbf{Sch})/S)}_{\text{fppf}}$. Conversely, given $\mathscr{H} \in \widetilde{((\mathbf{Sch})/S)}_{\text{fppf}}$ sheaf of $\mathcal{O}$-mods, and $T \xrightarrow{f} S$, we get $\mathscr{H}_T \in (\mathbf{Qcoh})(T)$ defined by

$$\mathscr{H}_T(U \subseteq T) = \mathscr{H}(U)$$

By construction, $\mathscr{F} \in (\mathbf{Qcoh})(S)$ is given by $(\mathscr{F}_{big})_S = \mathscr{F}$.

Proposition 3.1.12

Let $\mathscr{F} \in (\mathbf{Qcoh})(S)$, $\mathscr{G}$ a fppf sheaf of $\mathscr{O}$-mods. Then

$$\mathrm{Hom}_{\mathscr{O}}(\mathscr{F}_{big}, \mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{G}_S)$$

given by $\alpha \mapsto \alpha|_{S_{Zar}}$.

Proof. Exercise: it is enough to check Zariski local on S, so we can assume S is affine. Then we get

$$\mathscr{F}_2 := \mathscr{O}_S^J \to \mathscr{F}_1 := \mathscr{O}_S^I \to \mathscr{F} \to 0$$

where I, J are index sets. Then the functor f^* is right exact, so $\mathscr{F}_{2,big} \to \mathscr{F}_{1,big} \to \mathscr{F}_{big} \to 0$. As a result, we get

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathscr{F}_{big}, \mathscr{G}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathscr{F}_{1,big}, \mathscr{G}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathscr{F}_{2,big}, \mathscr{G}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\mathscr{F}, \mathscr{G}_S) \arrow[r] & \mathrm{Hom}(\mathscr{F}_1, \mathscr{G}_S) \arrow[r] & \mathrm{Hom}(\mathscr{F}_2, \mathscr{G}_S) \arrow[r] & 0
\end{tikzcd}$$

So, we can assume $\mathscr{F} = \mathscr{O}_S^I$. Therefore, we may assume $\mathscr{F} = \mathscr{O}_S$, i.e. $|I| = 1$. Now, observe elements in $\mathrm{Hom}_{\mathscr{O}}(\mathscr{O}, \mathscr{G})$ means that we have compatible maps $\phi_T : \mathscr{O}(T \to S) \to \mathscr{G}(T \to S)$. In particular, this means for any f with diagram

$$\begin{tikzcd}
T \arrow[r, "f"] \arrow[dr, "f"] & S \arrow[d] \\
& S
\end{tikzcd}$$

we get $f : (T \to S) \to (S \to S)$. Thus we get diagram

$$\begin{tikzcd}
\mathscr{O}(S) \arrow[r, "\phi_S"] \arrow[d, "f^*"] & \mathscr{G}(S) \arrow[d, "f^*"] \\
\mathscr{O}(T) \arrow[r, "\phi_T"] & \mathscr{G}(T)
\end{tikzcd}$$

so ϕ_T is determined by ϕ_S. Thus the map $\mathrm{Hom}_{\mathscr{O}}(\mathscr{O}, \mathscr{G}) \xrightarrow{\eta} \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{O}_S, \mathscr{G}_S)$ is isomorphism and hence ϕ_S iff $\xi \in \mathscr{G}(S)$ then $\phi_T(1) = f^*\xi$.

Definition 3.1.13

A ***big quasi-coherent sheaf on*** S of $\mathscr{O}$-mods is $\mathscr{F}$ on S_{fppf} such that:

1. $\forall T \to S$, $\mathscr{F}_T := \mathscr{F}|_{T_{Zar}} \in (\mathbf{Qcoh})(T)$
2. $\forall T \xrightarrow{f} S$, $\mathscr{F}_T \to f^*\mathscr{F}_S$ is an isomorphism.

Proposition 3.1.14

THere is an equivalent of categories

$$\begin{gathered}(\mathbf{Qcoh})(S_{Zar}) \xrightarrow{\sim} (\mathbf{Qcoh})(S_{fppf}) \\ \mathscr{F} \mapsto \mathscr{F}_{big} \\ \mathscr{G}_S \leftarrow\!\shortmid \mathscr{G}\end{gathered}$$

We get fibered category $(\mathbf{Qcoh}) \to \mathcal{C}$.

Theorem 3.1.15

If we have fppf $f : Y \twoheadrightarrow X$ *then* $(\mathbf{Qcoh})(X) \xrightarrow{\sim} (\mathbf{Qcoh})(Y \to X)$.

Proof. We only show the local case and the full proof can be found in the book.

Martin reduces to the case where $f : Y \to X$ is qcqs (quasi-compact, quasi-separated). In this case, pushforward of quasi-coherent sheaf is quasi-coherent, i.e. $f(\text{qcoh}) = \text{qcoh}$.

Then, we note

$$(\mathbf{Sh})(Y \xrightarrow{f} X) \underset{\eta}{\xrightarrow{\sim}} (\mathbf{Sh})(X)$$

$$(\mathscr{F}, \sigma) \mapsto \text{Equalizer of } f_*'S$$

Since $f_*(\text{qcoh}) = \text{qcoh}$, and equalizer of qcoh is qcoh, we see η sends $(\mathbf{Qcoh})(Y \to X)$ to $(\mathbf{Qcoh})(X)$.

In what follows, let's do some examples. In particular, let's start with what we already know, but in this new language.

First, let's do descents for closed subschemes.

Proposition 3.1.16

Suppose we have fppf cover

$$X \times_Y X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \underset{fppf}{\overset{f}{\twoheadrightarrow}} Y$$

Then the set of closed $W \subseteq Y$ *is equivalent to the set of closed* $Z \subseteq X$ *such that* $p_1^* Z = p_2^* Z$ *given by the map* $W \mapsto f^{-1}(W)$.

Proof. We note closed $W \subseteq Y$ is the same as quasi-coherent sheaf of ideals $\mathscr{I}_W \subseteq \mathscr{O}_Y$. Because f, p_1, p_2 are flat, pullback of ideal is an ideal. The result follows from descent of quasi-coherent sheaves applied to ideal sheaves.

In a similar manner, we have descents for open embeddings.

Indeed, let $(\mathbf{Op})$ be the category whose objects are pairs $(X, U \subseteq X)$ where X is a scheme and U an open subscheme of X. A morphism $(X', U') \to (X, U)$ is a morphism of schemes $f : X' \to X$ such that $U' \subseteq f^{-1}(U)$.

Then we get a functor

$$p : (\mathbf{Op}) \to (\mathbf{Sch}), \quad p(X, U) = X$$

This makes $(\mathbf{Op})$ into a fibered category over schemes.

Proposition 3.1.17

Any fppf covering $f : S' \to S$ is a effective descent for $(\mathbf{Op})$.

Proof. As usual, let $S'' := S' \times_S S'$, we need to show if $U' \subseteq S'$ with $p_1^{-1}(U') = p_2^{-1}(U')$ then $U' = f^{-1}(U)$ for some unique open subscheme $U \subseteq S$.

Well, the uniqueness is clear as $S' \to S$ is surjective. It remains to show existence. But observe since $f : S' \to S$ is fppf in particular it is a open map. Hence $U := f(U')$ is open in S, and it remains to show $f^{-1}(U) \subseteq U'$, as $U' \subseteq f^{-1}(U)$ is trivial. Pick $x \in f^{-1}(U)$ that is not in U', then we can find $y \in U'$ such that $f(x) = f(y)$ as $U = f(U')$ and $f(x) \in U$. But such pair (x, y) defines a point in S'' that lies in $p_2^{-1}(U)$ but not in $p_1^{-1}(U)$ (as $x \notin U'$). This is a contradiction and so $U' = f^{-1}(U)$ as desired.

In a very similar manner, we get descent for affine maps.

Let $(\mathbf{Aff}) \to (\mathbf{Sch})$ be the fibered category with objects $(X' \xrightarrow{g} X, X)$ with g affine map.

Proposition 3.1.18

If we have fppf cover $S' \twoheadrightarrow S$, then $(\mathbf{Aff})_S \xrightarrow{\sim} (\mathbf{Aff})(S' \to S)$.

Proof. Say $X \to S$ is affine, this is the same as $X = \operatorname{Spec} \mathscr{A}$ where $\mathscr{A}$ is qcoh sheaf of $\mathscr{O}_S$-algebra. Then we have descent for quasi-coherent sheaves, and we want to make sure it is still $\mathscr{O}_S$-algebra.

That is, how do we know if $\mathscr{A}' = f^*\mathscr{A}$ and $\mathscr{A}'$ a $\mathscr{O}_{S'}$-algebra, then $\mathscr{A}$ is $\mathscr{O}_S$-algebra?

Being an algebra means we have $m : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ with commutativity diagrams. But this is a map and hence $m' : \mathscr{A}' \times \mathscr{A}' \to \mathscr{A}'$ descents to get m and diagrams can be checked locally.

Of course why stop here when we can go quasi-affine? Recall we say $f: X \to Y$ is quasi-affine if there exists factorization

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & W \arrow[d, "g"] \\ & Y \end{tikzcd}$$

where j quasi-compact open imbedding and g affine. Equivalently, we say f is quasi-affine if the inverse image of every affine open of Y is quasi-affine in X, where a quasi-affine scheme means it is quasi-compact and isomorphic to an open subscheme of an affine scheme, i.e. X is quasi-affine scheme iff the canonical morphism $X \to \operatorname{Spec}(\Gamma(X, \mathscr{O}_X))$ is quasi-compact open immersion.

Now let (**QAff**) be the category with objects being quasi-affine $f: X \to Y$ and morphisms are commutative squares and $p: (\mathbf{QAff}) \to (\mathbf{Sch})$ be the functor sending $f: X \to Y$ to Y.

Proposition 3.1.19

Any fppf cover $S' \to S$ is an effective descent for (**QAff**).

Proof. Any quasi-affine $f: X \to Y$ admits $X \xrightarrow{j} Z \xrightarrow{g} Y$ where j is open imbedding and g is affine. In particular, if we let $\mathscr{A}$ be the quasi-coherent sheaf of algebras $f_* \mathscr{O}_X$, then Z is just $\operatorname{Spec}_Y(\mathscr{A})$ and j is the canonical map. Observe this construction commutes with flat base change on Y.

The key for this proof is just combine descents for open embedding and affine morphisms.

Now consider the functor

$$\epsilon : (\mathbf{QAff})(S) \to (\mathbf{QAff})(S' \to S)$$

Since schemes are fppf sheaves, this functor is fully faithful. Indeed, suppose we have $X_i \to S$ for $i = 1, 2$. Then for any $T \to S$, giving a T-morphism $(X_1)_T \to (X_2)_T$ is the same as giving an element in $h_{X_2}((X_1)_T)$ where h_{X_2} is the functor represents X_2. In particular, we get exact sequence

$$\operatorname{Hom}_S(X_1, X_2) \longrightarrow \operatorname{Hom}_{S'}((X_1)_{S'}, (X_2)_{S'}) \rightrightarrows \operatorname{Hom}_{S''}((X_1)_{S''}, (X_2)_{S''})$$

where $S'' = S' \times_S S'$. Next exchange the role of X_1 and X_2 we get fully faithful of the functor as desired.

It remaisn to show essentially surjective. That is, given any $(f': X' \to S', \sigma) \in (\mathbf{QAff})(S' \to S)$ we need to find a quasi-affine $X \to S \in (\mathbf{QAff})(S)$ that maps to this pair.

Well, given $f' : X' \to S'$ we get factorization $X' \xrightarrow{j'} Z' \xrightarrow{g'} S'$, and since this factorization commutes with flat base change, σ induces a unique isomorphism

$$\tilde{\sigma} : p_1^* Z' \to p_2^* Z'$$

such that the diagram of S''-schemes

$$\begin{tikzcd}
p_1^*X' \arrow[r, "\sigma"] \arrow[d] & p_2^*X' \arrow[d] \\
p_1^*Z' \arrow[r, "\tilde{\sigma}"] & p_2^*Z'
\end{tikzcd}$$

commutes. Also, since σ satisfies cocycle condition, $\tilde{\sigma}$ satisfies cocycle condition. Thus by descent for affine morphisms we showed above, $(Z', \tilde{\sigma})$ is induced by affine $Z \to S$. Now note X' defines an object of $(\mathbf{Op})(Z' \to Z)$, and thus it is the pullback of a unique open subset $X \hookrightarrow Z$. This open embedding is quasi-compact as it is pullback to Z' is quasi-compact. This object $X \to Z \to S$ is what we are looking for, and we are done.

Now recall $\mathcal{M}_g \to (\mathbf{Sch})_{\text{ét}}$, the fibered category of families of smooth curves of genus g, i.e. $C \to S \in \mathcal{M}_g(S)$ iff $C \to S$ is smooth and proper scheme morphism of finite presentation, such that every geometric fiber is connected curve of genus g

Proposition 3.1.20

If $g \geq 2$ then $\mathscr{M}_g$ is a stack over $(\mathbf{Sch})_{\text{ét}}$.

Proof. To prove this, we will use the alternative description of stacks means object and morphisms glue.

To show morphisms glue, we want to show that, for $C \to S$ and $D \to S$ of genus g and commutative diagrams

$$\begin{tikzcd}
C_{S_{ij}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_{S_i} \arrow[r] \arrow[d] \arrow[rr, bend left, "f_i"] \arrow[dr, phantom, "\square"] & C \arrow[r, dashed, "\exists! f"] \arrow[d] & D \\
S_{ij} \arrow[r] & S_i \arrow[r] & S \arrow[ur] &
\end{tikzcd}$$

of solid arrows, we have unique arrow f make the diagram commute. However, this follows from fppf and hence étale descent.

Next we show objects glue. This is the same as to show, given diagram

$$\begin{tikzcd}
C_i|_{S_{ij}} \arrow[r, "\alpha_{ij}"] \arrow[dr] \arrow[rr, bend left] & C_j|_{S_{ij}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_j \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, dashed] \\
& S_{ij} \arrow[r] & S_j \arrow[r] & S
\end{tikzcd}$$

for all i, j with $\pi_i : C_i \to S_i$ smooth curves of genus g, and α_{ij} isomorphisms satisfying cocycle condition, we need to show there exists C make the above diagram commutative, and we have isomorphism $\phi_i : C|_{S_i} \to C_i$ so $\alpha_{ij} \circ \phi_i|_{C_{S_{ij}}} = \phi_j|_{C_{S_{ij}}}$.

We use the following fact: for $\pi : C \to S$, $\omega_{C/S}^{\otimes 3}$ is relatively very ample, where ω is the relative sheaf of differentials, and $F := \pi_* \omega_{C/S}^{\otimes 3}$ is a vector bundle of rank $5(g-1)$. In particular, $\omega_{C/S}^{\otimes 3}$ gives closed immersion $C \to \mathbb{P}(F)$ over S.

Thus, set $E_i = (\pi_i)_*(\omega_{C_i/S})$, we get closed immersion $C_i \to \mathbb{P}(E_i)$ over S_i. The isomorphisms α_{ij} induce isomorphisms $\beta_{ij} : E_i|_{S_{ij}} \to E_j|_{S_{ij}}$ satisfying the cocycle condition $\beta_{ik} \circ \beta_{ij} = \beta_{ik}$ on S_{ijk}. Descent for quasi-coherent sheaves implies there is a quasi-coherent sheave E on S and isomorphisms $\Psi_i : E|_{S_i} \to E_i$ such that $\beta_{ij} \circ \Psi_i|_{S_{ij}} = \Psi_j|_{S_{ij}}$. By descent we see E is a vector bundle.

Since the preimage of $C_i \subseteq \mathbb{P}(E_i)$ and $C_j \subseteq \mathbb{P}(E_j)$ inside $\mathbb{P}(E_{ij})$ are equal, it follows from descent for closed subschemes that there is $C \to S$ and isomorphisms ϕ_i so $\alpha_{ij} \circ \phi_i|_{C_{S_{ij}}} = \phi_j|_{C_{S_{ij}}}$. We see $C \to S$ is smooth and proper as those properties are étale-local on the target, and the fiber of $C \to S$ are connected genus g curves as the fibers of $C_i \to S_i$ are.

3.2 G-Bundles

In this section, we are going to start with G-bundles/torsors.

Before we actually introduce the object, let's talk a little bit about motivation of where it is been used. In particular, one of the most important examples of stacks are quotient stacks $[X/G]$, which is obtained from action of smooth algebraic group G on scheme X. The geometry of $[X/G]$ is, as you would imagine, just G-equivariant geometry of X.

However, to define $[X/G]$, we note the functor $(\mathbf{Sch}) \to (\mathbf{Sets})$ given by $S \mapsto X(S)/G(S)$ is not a sheaf, even the action is free. Thus, in order to define $[X/G]$, we need some other functors/a better notion of orbits.

For simplicity, let G and X be both over $\mathbb{C}$. For $x \in X(\mathbb{C})$, there is a G-morphism $\sigma_x : G \to X$ given by $g \mapsto g \cdot x$. Two points x, x' of X are in the same G-orbit if and only if there is a G-morphism $\phi : G \to G$ such that $\sigma = \sigma_{x'} \circ \phi$.

This can be done more generally, i.e. given a T-point $f : T \to X$, we can consider

$$\begin{tikzcd} G \times T \arrow[r] \arrow[d] & X \\ T \end{tikzcd}$$

where $G \times T \to X$ is given by $(g, t) \mapsto g \cdot f(t)$, and $G \times T \to T$ is just projection. Observe $G \times T \to X$ is a G-morphism. If we define our fibered category as those $T \to X$ objects, then it will not be an algebraic stack in the end, as it will not glue. To fix the problem, we need to replace the family $T \to X$ by what's called principle G-bundles, and we will focus on those objects in this section.

Thus, let X be a scheme and $\mathcal{C} = ((\mathbf{Sch})/X)_{\text{fppf}}$. We are going to start with a group scheme over X, say G, which is flat and locally of finite presentation over X.

Definition 3.2.1

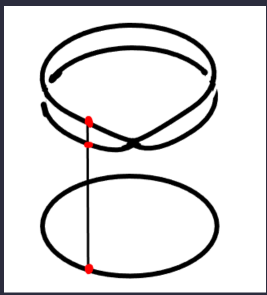
A ***(principle) G-bundle*** is a scheme $\pi : P \to X$ over X, where π is fppf, with an G-action $\rho : G \times P \to P$, such that the map $G \times_X P \xrightarrow{\sim} P \times_X P$ given by $(g, p) \mapsto (gp, p)$, is isomorphism, where $gp = \rho(g, p)$.

This $G \times_X P \xrightarrow{\sim} P \times_X P$ condition is equivalent to saying if $Y \to X$ and $P(Y) \neq \emptyset$, then the group action of $G(Y)$ acts on $P(Y)$ is simple and transitive, i.e. $G(Y)$ acts on $P(Y)$ has no stabilizers and for all $p, p' \in P(Y)$ there exists $g \in G(Y)$ so $gp = p'$, i.e. for all $p, p' \in P(Y)$ there exists unique $g \in G(Y)$ so $p' = gp$.

So, before we give examples, we talk about the idea of G-bundle. In particular, we can think of P as a group without a choice of identity. Here, if we give two elements, we only care about the difference, not which particular g we are working with(so it is similar to the idea of potential functions in physics, i.e. we only care about diff of potentials, not the initial value).

Example 3.2.2

Consider $\mathbb{C} \supseteq S^1 \to S^1$ given by $z \mapsto z^2$. Here is a picture:



Then there is a $\mathbb{Z}/2\mathbb{Z} = S^1 \coprod S^1$ action: we swap the 2 strands, i.e. $z \mapsto -z$. Locally, P is $S^1 \coprod S^1 = \mathbb{Z}/2$ but not globally.

Definition 3.2.3

A ***map of G-bundles*** $P \to P'$ is a map of S-schemes $f : P \to P'$ such that

$$\begin{tikzcd} G \times P \arrow[r, "\rho"] \arrow[d, "\mathrm{Id} \times f"'] & P \arrow[d, "f"] \\ G \times P' \arrow[r, "\rho'"] & P' \end{tikzcd}$$

Definition 3.2.4

Let $\mathcal{C}$ be a site and μ a sheaf of groups. A ***μ-torsor*** is a sheaf $\mathscr{P}$ with $\mu \times \mathscr{P} \to \mathscr{P}$ such that:

1. for all $X \in \mathcal{C}$, there exists $\{X_i \to X\} \in \mathrm{Cov}(X)$ such that $\mathscr{P}(X_i) \neq \emptyset$ for all i.
2. we have $\mu \times \mathscr{P} \xrightarrow{\sim} \mathscr{P} \times \mathscr{P}$ where the map is given by $(g, p) \mapsto (gp, p)$.

Definition 3.2.5

A μ-torsor $\mathscr{P}$ is ***trivial*** if $\mathscr{P} \cong \mu$ as μ-torsor.

Proposition 3.2.6

If μ is representable on $\mathcal{C} = ((\mathbf{Sch})/X)_{fppf}$ by a group scheme G, then

$$\begin{aligned} \{\text{Principle } G\text{-bundle}\} &\xrightarrow{\epsilon} \{\mu\text{-torsor}\} \\ P &\mapsto h_P \end{aligned}$$

is fully faithful. And if $G \to X$ is affine then ϵ is equivalence.

Proof. Yoneda says $P \mapsto h_P$ is fully faithful. Why h_P is μ-torsor? In other word, in def of torsor, (2) holds by definition, but why we have (1)?

Consider

$$\begin{tikzcd} & P \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

We want fppf cover $Y' \twoheadrightarrow Y$ such that $P(Y') \neq \emptyset$. However, note we have

$$\begin{tikzcd} Y' = P_Y \arrow[r, "p \in P(Y')"] \arrow[d, two heads, "\text{fppf}"] \arrow[dr, phantom, "\square"] & P \arrow[d, two heads, "\text{fppf}"] \\ Y \arrow[r] & X \end{tikzcd}$$

Hence we are done.

Now assume $G \to X$ is affine and $\mathscr{P}/X$ is μ-torsor, we want $\mathscr{P} = h_P$ where P is a G-bundle.

Note by assumption, we can find fppf cover $\{X_i \twoheadrightarrow X\}$ such that $\mathscr{P}(X_i) \neq \emptyset$. But then $\mathscr{P}|_{X_i} \cong \mu_{X_i}$ since $\mu(X_i)$ acts on $\mathscr{P}(X_i)$ simple and transitive where $\mu_{X_i} = h_{G \times_X X_i}$, i.e. $\mathscr{P}$ is locally representable.

Since $\mathscr{P}$ is a sheaf over X, $\mathscr{P}|_{X_i}$ have canonical descent data. On the other hand, $\mathscr{P}|_{X_i} \cong G \times_X X_i$ is scheme affine over X_i. So, this yields descent data for the $G \times_X X_i$. But result we showed above says descent data is effective for affine morphisms. Therefore, we get a scheme $P \to X$ that also defines a sheaf h_P which must agree with $\mathscr{P}$ because they are sheaves with same descent data.

Last thing we need to check is that why $P \to X$ is a G-bundle.

We need to check there exists action $G \times P \to P$ such that the graph of the action $G \times_X P \xrightarrow{\sim} P \times_X P$ is isomorphism.

Why $G \times_X P \to P$ exists? Well, we have $G \times P \to P$ by Yoneda because we have $\mu \times \mathscr{P} \to \mathscr{P}$ action such that $\mu \times \mathscr{P} \xrightarrow{\sim} \mathscr{P} \times \mathscr{P}$ so by Yoneda we get $G \times_X P \to P$ with $G \times P \xrightarrow{P} \times P$.

Remark 3.2.7

If $G \to X$ is smooth, then $P \to X$ is also smooth. This is by fppf descent, i.e. we get

$$\begin{tikzcd} G \times_X P \arrow[r, "\cong"] \arrow[dr, "smooth"'] & P \times_X P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & P \arrow[d, two heads, "\mathrm{fppf}"] \\ & P \arrow[r, two heads, "\mathrm{fppf}"] & X \end{tikzcd}$$

and hence the arrow $P \to X$ is smooth as well (smooth can be verified over an fppf covering where P is trivial). Note here smooth does nothing and it can be changed to basically any property (that descent).

Example 3.2.8

Let X be a scheme and n an integer invertible on X (i.e. n is invertible in $\mathscr{O}_X(X)$). Let μ_n be the group scheme with $\mu_n(S) = \{f \in \mathscr{O}_S^* : f^n = 1\}$. In this example we describe the category of μ_n-torsors $(\mathbf{Tors})_{\mu_n}$ on the étale site of X.

Let Σ_n be the category of pairs (L, σ) where L is an line bundle on X and $\sigma : L^{\otimes n} \to \mathscr{O}_X$ a trivialization of the nth power of L. Note that this L can be both considered as a sheaf in Zariski or étale topology. A morphism $(L, \sigma) \to (L', \sigma')$ is an isomorphism $\rho : L \to L'$ with $\rho^{\otimes n} \circ \sigma' = \sigma$.

Now we define a functor

$$F : \Sigma_n \to (\mathbf{Tors})_{\mu_n}$$

as follows. For (L, Σ) let $\mathscr{P}_{(L,\Sigma)}$ be the sheaf on the étale site of X associating to any $U \to X$ the set of trivializations $\lambda : \mathscr{O}_U \to L|_U$ such that the composition

$$\mathscr{O}_U \xrightarrow{\lambda^{\otimes n}} L^{\otimes n}|_U \xrightarrow{\sigma} \mathscr{O}_U$$

is the identity map. There is an action of $\mu_n(U)$ on $\mathscr{P}_{(L,u)}(U)$ defined by $\zeta \in \mu_n(U)$ sends λ to $\zeta \cdot \lambda$. This action is simply transitive on $\mathscr{P}_{(L,\Sigma)}$ and so $\mathscr{P}_{(L,\Sigma)}$ is a μ_n-torsor.

We remark that it is essential we work with étale topology here. In general we get trivialization $\tau : \mathscr{O}_X \to L$ on Zariski topology, which imply the composite map $\mathcal{O}_X \to L^{\otimes n} \to \mathcal{O}_X$ is just multiplication by $f \in \mathcal{O}_X^*$. But then to find nth root of f, we need to pass to étale cover.

On the other hand, we can define $G : (\mathbf{Tors})_{\mu_n} \to \Sigma_n$ in the following way. Let $\mathcal{P}$ be μ_n-torsor, let $L_{\mathcal{P}}$ be the line bundle corresponding to the $\mathcal{O}_X^*$-torsor $T_{\mathcal{P}}$, where for étale cover $U \to X$, $T_{\mathcal{P}}(U \to X)$ is the set of morphisms of sheaves of sets $\mathcal{P}|_U \to \mathcal{O}_U^*$ which commutes with the action of μ_n, where μ_n acts on $\mathcal{O}_U^*$ by multiplication. Now if $\mathcal{P}_U$ is trivial and $p \in \mathcal{P}(U)$ a section, then we get trivialization of $T_{\mathcal{P}}$ from the unique map $\mathcal{P}|_U \to \mathcal{O}_U^*$ sending p to 1. If p' is another trivialization obtained by μ_n-action on p, then the trivialization of $T_{\mathcal{P}}$ differ by a μ_n-action, i.e. we get canonical trivialization $\sigma_{\mathcal{P}}$ of $L_{\mathcal{P}}^{\otimes n}$, and the association $\mathcal{P} \mapsto (L_{\mathcal{P}}, \sigma_{\mathcal{P}})$ defines our functor G.

We will not show F, G are quasi-inverse of each other.

Proposition 3.2.9

Let X be a scheme, $\mathcal{F}$ be a sheaf on X, and $\mu = \mathrm{Aut}(\mathcal{F})$. Then there is an equivalence of categories

$$\{\mu-\text{torsors on } X\} \xrightarrow{\sim} \left\{\begin{matrix}\text{sheaves locally} \\ \text{isomorphic to } \mathcal{F}\end{matrix}\right\}$$

given by map

$$\mathcal{P} \mapsto \mathcal{H}_{\mathcal{P}} := \mathrm{Hom}_{\mu}(\mathcal{P}, \mathcal{F})$$

$$\mathcal{P}_{\mathcal{H}} := \mathrm{Isom}(\mathcal{F}, \mathcal{H}) \leftarrow\!\shortmid \mathcal{H}$$

where Hom_{μ} means μ-equivariant maps.

Proof. We first show $\mathcal{P}_{\mathcal{H}}$ is μ-torsor. We deinfe a map $\mu \times \mathcal{P}_{\mathcal{H}} \to \mathcal{P}_{\mathcal{H}}$ by $\mu(Y) \times \mathcal{P}_{\mathcal{H}}(Y) \to \mathcal{P}_{\mathcal{H}}(Y)$ by $(f, \lambda) \mapsto f \circ \lambda$, i.e. we get a diagram

$$\begin{tikzcd} \mathcal{F}_Y \arrow[r, "f", "\sim"'] \arrow[dr, dashed, "f\circ\lambda"'] & \mathcal{F}_Y \arrow[d, "\lambda"', "\sim"] \\ & \mathcal{H}_Y \end{tikzcd}$$

Why is this simple and transitive? If $\mathcal{P}_{\mathcal{H}}(Y) \neq \emptyset$, then for $\lambda, \lambda' \in \mathcal{P}_{\mathcal{H}}(Y)$ we get

$$\begin{tikzcd} \mathcal{F}_Y \arrow[r, "f", "\sim"'] \arrow[dr, "p"'] & \mathcal{H}_Y \arrow[d, "(\lambda')^{-1}"', "\sim"] \\ & \mathcal{F}_Y \end{tikzcd}$$

i.e. $\lambda' = p \circ \lambda$ for a unique p.

Why $\mathcal{H}_{\mathcal{P}}$ is a sheaf? Given

$$Y'' = Y' \times_Y Y' \underset{p_2}{\overset{p_1}{\rightrightarrows}} Y' \underset{f}{\overset{\text{fppf}}{\twoheadrightarrow}} Y$$

Then for $\phi' \in \mathscr{P}_{\mathscr{H}}(Y')$ with $\phi' : \mathscr{F}_{Y'} \to \mathscr{H}_{Y'}$ μ-equivariant map such that $p_1^*\phi' = p_2^*\phi'$, since sheaves satisfy descent so we get a unique $\phi : \mathscr{F}_Y \to \mathscr{H}_Y$ such that $f^*\phi = \phi'$.

Why ϕ is μ-equivariant? Viz, we want commutative diagram

$$\begin{tikzcd} \mu \times \mathscr{F}_Y \arrow[r] \arrow[d, "\mathrm{Id} \times \phi"] & \mathscr{F}_Y \arrow[d, "\phi"] \\ \mu \times \mathscr{H}_Y \arrow[r] & \mathscr{H}_Y \end{tikzcd}$$

However, diagram commutes fppf locally over Y', hence it commutes over Y.

Why $\mathscr{H}_{\mathscr{P}} \cong \mathscr{F}$ locally?

Locally we get μ-equivariant isomorphism $\mathscr{P} \cong \mu$. So we claim $\mathscr{H}_{\mathscr{P}}$ is locally isomorphic to $\mathrm{Hom}_\mu(\mu, \mathscr{F})$. Well, suppose we have $\mu \xrightarrow{\alpha} \mathscr{F}$. Then we see $\alpha(\zeta) = \alpha(\zeta \cdot 1) = \zeta \cdot \alpha(1)$ and hence $\mathrm{Hom}_\mu(\mu, \mathscr{F}) \cong \mathscr{F}$ given by $\alpha \mapsto \alpha(1)$. This concludes the proof.

Remark 3.2.10

If $\mathscr{F}$ is an $\mathscr{O}_X$-module, and $\mu = \mathrm{Aut}_{\mathscr{O}_X}(\mathscr{F})$, then μ-torsor are equivalent to $\mathscr{O}_X$-modules $\mathscr{H}$ which are locally isomorphic to $\mathscr{F}$ as $\mathscr{O}_X$-modules.

Example 3.2.11

Take $\mathscr{F} = \mathscr{O}_X$ as $\mathscr{O}_X$-modules. Then we are shown that line bundles are isomorphic to μ-torsors, where $\mu = \mathrm{Aut}_{\mathscr{O}_X}(\mathscr{O}_X) = \mathscr{O}_X^*$, where $\mathscr{O}_X^*$ is representable by $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[x, x^{-1}]$. Hence, we see the line bundles are the same as $\mathbb{G}_m$-bundles.

Clearly we don't have to restrict to $\mathscr{O}_X$. In other word, we can take $\mathscr{F} = \mathscr{O}_X^n$, then $\mu = \mathrm{GL}_n(\mathscr{O}_X)$, which is representable by scheme $\mathrm{GL}_n = \operatorname{Spec} \mathbb{Z}[x_{ij}, x_{ij}^{-1} : 1 \leq i, j \leq n]$. We can also define this as $M_n := \operatorname{Spec} \mathbb{Z}[x_{ij}, 1 \leq i, j \leq n]$ and then $\deg(x_{ij})$ is a polynomial and we look at $\mathrm{GL}_n := M_n \backslash V(\det)$, which is affine. In this case, we just get rank n vector bundles are equivalent to GL_n-bundles.

Example 3.2.12

We can also talk about Brauer-Sever varieties, which are locally isomorphic to $\mathbb{P}^n$. Those are PGL_n-torsors, and it is related to Azumaya algebras and the Brauer groups, where the Brauer groups are deeply related to class field theory in number theory.

Chapter 4

Algebraic Stacks

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今古恨，几千般，只应离合是悲欢？江头未是风波恶，别有人间行路难。

辛弃疾

4.1 Algebraic Space

Recall that stack is a category fibered in groupoids where descent holds for all covering maps.

Proposition 4.1.1

If $\mathcal{F} \to \mathcal{C}$ is a stack, then for all $X \in \mathcal{C}$ and $x, y \in \mathcal{F}(X)$, we have $\operatorname{Isom}(x, y)$ is a sheaf on $\mathcal{C}/X$.

Proof. If $Y' \xrightarrow{f} Y$ is a covering, then consider

$$Y'' = Y' \times_Y Y' \overset{p_2}{\underset{p_1}{\rightrightarrows}} Y' \xrightarrow{f} Y$$

in $\mathcal{C}/X$, i.e. we also have an arrow $Y \to X$ by default. Then we get $x, y \in X$ and hence we get a bunch of pullbacks

$$\begin{tikzcd} Y'' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & Y' \arrow[r, "f"] & Y \arrow[r, "g"] & X \end{tikzcd}$$

$$\begin{tikzcd} p_1^* f^* g^* x \arrow[d, shift right, "p_1^*\phi"'] \arrow[d, shift left, "p_2^*\phi"] & f^* g^* x \arrow[d, "\phi"] & g^* x & x \\ p_1^* f^* g^* y & f^* g^* y & g^* y & y \end{tikzcd}$$

Note here the double arrows of pullback is actually a square

$$\begin{tikzcd} p_1^* f^* g^* x \arrow[r, "\sim", "\text{can}"'] \arrow[d, "p_1^*\phi"] & p_2^* f^* g^* x \arrow[d, "p_2^*\phi"] \\ p_1^* f^* g^* y \arrow[r, "\sim", "\text{can}"'] & p_2^* f^* g^* y \end{tikzcd}$$

Suppose $p_1^*\phi = p_2^*\phi$. We want ϕ to descent to $g^*x \xrightarrow{\sim} g^*y$. By definition, ϕ defines an isomorphism $(f^*g^*x, \sigma_{\text{can}}) \xrightarrow{\sim} (f^*g^*y, \sigma_{\text{can}})$ in $\mathcal{F}(Y' \to Y)$. However, we know $\mathcal{F}(Y) \underset{\epsilon}{\xrightarrow{\sim}} \mathcal{F}(Y' \to Y)$ as $\mathcal{F}$ is a stack, thus we are done as desired.

Proposition 4.1.2

Let

$$\begin{tikzcd} & \mathcal{F}_1 \arrow[d] \\ \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \end{tikzcd}$$

be maps of stacks over $\mathcal{C}$. Let $\mathcal{F} := \mathcal{F}_1 \times_{\mathcal{F}_3} \mathcal{F}_2$ be the fibered product of category fibered in groupoids. Then $\mathcal{F}$ is a stack.

Proof. By definition, $(\mathcal{F}_1 \times_{\mathcal{F}_3} \mathcal{F}_2)(X) = \mathcal{F}_1(X) \times_{\mathcal{F}_3(X)} \mathcal{F}_2(X)$. Similarly, an easy check shows

$$\mathcal{F}(\{X_i \to X\}) = \mathcal{F}_1(\{X_i \to X\}) \times_{\mathcal{F}_3(\{X_i \to X\})} \mathcal{F}_2(\{X_i \to X\})$$

Each $\mathcal{F}_i(X) \xrightarrow{\sim} \mathcal{F}_i(\{X_i \to X\})$ and hence $\mathcal{F}$ is a stack.

Just like for sheaves we have sheafification, for stacks we have "stackification".

Theorem 4.1.3: Theorem 4.6.5 in Martin

Let $\mathcal{F}$ be category fibered in groupoids over $\mathcal{C}$. Then there exists stack $\mathcal{F}^a/\mathcal{C}$ and $\mathcal{F} \to \mathcal{F}^a$ such that for all stacks $\mathcal{H}/\mathcal{C}$, we have $\mathrm{HOM}_{\mathcal{C}}(\mathcal{F}^a, \mathcal{H}) \xrightarrow{\sim} \mathrm{HOM}_{\mathcal{C}}(\mathcal{F}, \mathcal{H})$.

Next, we are going to define algebraic spaces, but first, we give some ideas.

Idea: what is a scheme? A scheme is affine schemes glued in Zariski topology. Then algebraic space is affine schemes glued in étale topology.

Of course, now we are just begging for the question of what if fppf topology. It turns out, it is not so easy to answer this question. The answer is that a theorem of Artin, where he showed they are the same as algebraic spaces.

Let S be a scheme, $\mathcal{C} = ((\mathbf{Sch})/S)_{\text{ét}}$.

Definition 4.1.4

A morphism of sheaves $\mathcal{F} \to \mathcal{H}$ is ***representable by schemes*** if for all $T \to \mathcal{H}$ with $T = h_T$ scheme, the fibered product (as cat fibered in sets) $\mathcal{F} \times_{\mathcal{H}} T$ is a scheme.

So, when we say $\mathcal{F} \to \mathcal{H}$ is representable by schemes, we mean for all T we get the following diagram

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{H}} T \in (\mathbf{Sch}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{F} \arrow[d] \\ T \arrow[r] & \mathcal{H} \end{tikzcd}$$

Definition 4.1.5

Let P be a property of morphisms of schemes. If for all diagrams

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

we have:

1. f has P implies f' has P, then we say P is ***stable under base change***.
2. if g is a Zar (ét, sm, fppf) covering, then f has P iff f' has P, then we say P is ***local on the base for the Zar (ét, sm, fppf) topology***.

Finally, we say P is ***local on the source for the Zar (ét, sm, fppf) topology***, if for all diagrams

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

with π a Zar (ét, sm, fppf) covering, f has P iff g has P.

Definition 4.1.6

If $\mathcal{F} \xrightarrow{f} \mathcal{H}$ is representable by schemes and P is a property which is stable under base change and local on the base, then we say f ***has*** P iff for all $T \to \mathcal{H}$ with T schemes, $\mathcal{F} \times_{\mathcal{H}} T \to T$ has P.

We remark that if $\mathcal{F} = h_X$ and $\mathcal{H} = h_Y$ then $\mathcal{F} \to \mathcal{H}$ has P iff $X \to Y$ has P because we can consider the identity $Y \to \mathcal{H}$ map and pull it back using this.

Lemma 4.1.7

Let $\mathcal{F}$ be a presheaf on $(\mathbf{Sch})/S$. Then $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is representable by schemes if and only if $T \to \mathcal{F}$ is representable by schemes for all schemes T.

Proof. Consider

$$\begin{tikzcd} T \times_{\mathcal{F}} T' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d, "f"] \\ T' \arrow[r, "f'"] & \mathcal{F} \end{tikzcd}$$

Then we get

$$\begin{tikzcd} \mathcal{H} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \times T' \arrow[d, "{(f,f')}"] \\ \mathcal{F} \arrow[r, "\Delta"] & \mathcal{F} \times \mathcal{F} \end{tikzcd}$$

Here $\mathcal{H}$ is (x, t, t') with $(x, x) = \Delta(x) = (f(t), f'(t))$. So, $\mathcal{H} \cong T' \times_{\mathcal{F}} T$. Δ representable by schemes, so $\mathcal{H}$ is scheme, so $T' \times_{\mathcal{F}} T$ is scheme, as desired.

Q.E.D.

Definition 4.1.8

An ***algebraic space*** over S is a sheaf $\mathcal{F}$ for the big étale topology on $(\mathbf{Sch})/S$ such that:

1. $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is representable by scheme.
2. there exists S-scheme U and étale covering $\pi : U \twoheadrightarrow \mathcal{F}$

We note (2) makes sense because (1) implies π is representable by schemes and étale surjections are property that are stable under base change and local on the base.

Remark 4.1.9

Schemes are algebraic spaces because we see (1) is true as $\mathcal{F} = h_X$ is a scheme, and for (2) we choose $\pi = \mathrm{Id}_X$. This is because h_X is sheaf for fppf topology, so also it is also sheaf for étale topology.

Definition 4.1.10

A morphism $\mathfrak{X} \to \mathfrak{Y}$ of stacks is ***representable*** if for all diagrams with Y algebraic space

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ Y \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

we get X is an algebraic space.

Proposition 4.1.11

A morphism of stacks $\mathfrak{X} \to \mathfrak{Y}$ is representable iff for all diagrams with Y a scheme

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ Y \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

we get X is algebraic space.

Proof. If Y is algebraic space then we know X is a stack, which proves the forward direction. We show the converse. Hence, suppose we are given $Y \to \mathfrak{Y}$ where Y is a scheme, we want to show $\mathfrak{Z}$ is an algebraic space, where the stack $\mathfrak{Z}$ is defined by

$$\begin{tikzcd} \mathfrak{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ Y \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

(1): we want to show $\mathfrak{Z}$ is a sheaf. Since $\mathfrak{Z}$ is a stack, so we just need to show $\mathfrak{Z}$ is fibered in sets. Now consider this diagram

$$\begin{tikzcd} T' \arrow[r, two heads, "g"] \arrow[d, shift right, "x'"'] \arrow[d, shift left, "y'"] \arrow[dr, phantom, "\square"] & T \arrow[d, shift right, "x"'] \arrow[d, shift left, "y"] & \\ \mathfrak{Z}' \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ Y' \arrow[r, two heads, "et"] & Y \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

where we know Y' is a scheme, Y an algebraic space, and T a scheme. We have $x \underset{\phi}{\xrightarrow{\sim}} y$ in $\mathfrak{Z}(T)$ and we want ϕ to be the identity map. However, $\mathfrak{Z}'$ is algebraic space by hypothesis, T' is a scheme by the fact of Y being algebraic space plus Lemma 4.1.7. We get $g^*\phi : x' \xrightarrow{\sim} y'$ and $\mathfrak{Z}'$ is algebraic space, and in particular $\mathfrak{Z}'$ is a sheaf (and so its fibered in sets), so $g^*\phi = \mathrm{Id}$. Therefore g is étale covering and $\mathfrak{Z}$ is a stack, and $g^*\phi = \mathrm{Id}$, hence $\phi = \mathrm{Id}$ by descent.

(2): we want to show $\mathfrak{Z}$ has étale covering by scheme. We get sch $\twoheadrightarrow \mathfrak{Z}' \twoheadrightarrow \mathfrak{Z}$ where $\mathfrak{Z}'$ is algebraic space and all arrows are étale, hence we are done.

(3): It remains to show $\Delta_{\mathfrak{Z}}$ is representable by schemes. We do a diagram again:

$$\begin{tikzcd}
\mathfrak{Z}_T \arrow[r] \arrow[d, "\Delta_{\mathfrak{Z}_T}"'] \arrow[dr, phantom, "\square"] & \mathfrak{Z} \arrow[r] \arrow[d, "\Delta_{\mathfrak{Z}}"] & \mathfrak{X} \arrow[dd] \\
\mathfrak{Z}_T \times_T \mathfrak{Z}_T \arrow[r] \arrow[d, shift left] & \mathfrak{Z} \times_Y \mathfrak{Z} \arrow[d] \arrow[dr, phantom, "\square"] & \\
T \arrow[r] \arrow[u, shift left, "s"] \arrow[ur, dashed, "g"] & Y \arrow[r] & \mathfrak{Y}
\end{tikzcd}$$

where T is a scheme. This gives

$$\mathfrak{Z} \times_{\Delta, \mathfrak{Z} \times_Y \mathfrak{Z}, g} T = \mathfrak{Z}_T \times_{\Delta, \mathfrak{Z}_T \times_T \mathfrak{Z}_T, s} T =: W$$

where W is a scheme since $\mathfrak{Z}_T$ is an algebraic space. This concludes the proof.

So in the above, we proved various reduction techniques for verifying whether a space is algebraic space. Next, we are going to give a different characterizations of algebraic spaces.

Definition 4.1.12

Fix a base scheme S. An ***étale equivalence relation*** on S-scheme X is a monomorphism of schemes $R \hookrightarrow X \times_S X$ such that:

1. For every S-scheme T the T-points $R(T) \subseteq X(T) \times X(T)$ is an equivalence relation on $X(T)$.
2. The two maps $s, t : R \to X$ induced by the two projections are étale.

Note since R is an equivalence relation, we get an inclusion $X \hookrightarrow R$ induced by the diagonal $\Delta_X : X \to X \times_S X$.

Given an étale equivalence $R \hookleftarrow X \times_S X$, consider the presheaf

$$((\mathbf{Sch})/S)^{\mathrm{opp}} \to (\mathbf{Sets}), \quad T \mapsto X(T)/R(T)$$

We use X/R to denote the associated sheaf with respect to the étale topology on $((\mathbf{Sch})/S)^{\mathrm{opp}}$.

Proposition 4.1.13

1. *X/R is an algebraic space.*
2. *If Y is an algebraic space over S, $X \to Y$ an étale surjection with X a scheme, then $R := X \times_Y X \hookrightarrow X \times_S X$ is an étale equivalence. Moreover, $X/R \to Y$ is an isomorphism.*

Proof. Pretty complicated. We refer to Proposition 5.2.5 of Olsson's book.

Now let's talk about some examples of algebraic spaces. In the next section we will define algebraic stacks, and one of the most important class of algebraic stacks is quotient stacks. Let's first see consider a baby example of this (i.e. quotient algebraic spaces).

Example 4.1.14

Let X be a scheme, G a discrete group acting on X. Write $\rho : G \times X \to X$ the action, and assume this action is free, i.e. $j : G \times X \to X \times X$ given by $(g, x) \mapsto (x, \rho(g, x))$ is a monomorphism. Then, let X/G be the étale sheaf associated to the presheaf defined by $T \mapsto X(T)/G$. Then X/G is the quotient of X by the étale equivalence and thus it is an algebraic space.

We will spend a great deal of time in the next section to construct quotient stacks (and not invoke Proposition 4.1.13), but for now, let's see examples of X/G such that it is not a scheme. In other word, a scheme quotient by a group is not always a scheme!

Example 4.1.15: Donald Knutson

Let k be a field and consider $U = \operatorname{Spec} k[s,t]/(st)$ obtained by gluing two copies of the affine line along the origin. Let $U' \subseteq U$ be the open subset obtained by deleting the origin and set

$$R := U \coprod U'$$

Consider the two maps $\pi_1, \pi_2 : R \to U$ defined by the following. Both π_i restricts to identity on U, but on U' we define π_1 to be the natural inclusion and π_2 the map switches the two components.

Then

$$\pi_1 \times \pi_2 : R \to U \times U$$

is an étale equivalence and let $F = U/R$ be the resulting algebraic space. We claim F is not a scheme. Indeed, the map

$$s + t : U \to \mathbb{A}^1$$

is universal in the category of ringed spaces for maps from U which factor through

F (i.e. $R \rightrightarrows U \to \mathbb{A}^1$ is a coequalizer diagram). However, the induced map $F \to \mathbb{A}^1$ in the category of algebraic spaces is not an isomorphism, as the map $s+t$ is not étale.

Example 4.1.16

Let k be a field of characteristic 0 and $\mathbb{Z}$ acts on $\mathbb{A}^1_k$ by the translation action. This is a free action and the quotient $X := \mathbb{A}^1/\mathbb{Z}$ is an algebraic space that's not scheme.

Well, suppose its a scheme for contradiction. Then X has a surjective étale covering by $\mathbb{A}^1_k$ and thus X is smooth and connected, and for any affine open $U \subseteq X$ the sections $\mathscr{O}_X(U)$ are included into the $\mathbb{Z}$-invariants of the rational function field $\kappa(x)$, where $n \in \mathbb{Z}$ acts by $x \mapsto x+n$. The $\mathbb{Z}$-invariants in $k(x)$ are just the constant functions since a non-constant $\mathbb{Z}$-invariant $f \in k(x)$ would have infinitely many zeros and poles. Thus the coordinate ring of any non-empty open subset of X is k implying $X = \operatorname{Spec} k$. But this contradicts the fact $\mathbb{A}^1_k \to X$ is étale, and thus X does not contain any non-empty open subspaces which is a scheme.

Remark 4.1.17

Here are some random remarks:

1. One can show that every quasi-separated algebraic space has a dense open subspace which is a scheme. For a proof, see, e.g. Stack Project tag 06NN.
2. Let X be quasi-compact, quasi-separated algebraic space. If the functor $\Gamma(X,-)$ is eaxct on the category of quasi-coherent sheaves, then X is an affine scheme.
3. If X is a Noetherian algebraic space such that X_{red} is a scheme, then so is X.
4. A quasi-separated group algebraic space G locally of finite type over field k is a scheme.

4.2 Algebraic Stacks

We are now ready to define algebraic stacks.

Definition 4.2.1

Let P be property of morphisms which is local on the source for the étale topology, local on the target for the sm topology, and stable under base change. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a representable map of stacks over S. We say f ***has*** P if for all schemes T and diagram

$$\begin{tikzcd} \mathfrak{X}_T \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ T \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

we have g has property P, where we know $\mathfrak{X}_T$ is algebraic space.

Recall that, if $g : X \to T$ where T is scheme and X is algebraic space, then we say g has P if there exists étale covering $\tilde{X}$ with

$$\tilde{X} \xrightarrow[\pi]{et} X \xrightarrow{g} T$$

such that $g\pi$ has P.

Example 4.2.2

P could be étale, smooth, relative dim d, affine, finite, closed, immersion, open immersion, surjection, and so on. We note this list is smaller than the list for fppf descent as we require local on the source.

Definition 4.2.3

A stack $\mathfrak{X}/S$ is an ***algebraic/Artin stack*** if:

1. $\Delta_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is representable,
2. There exists scheme U so that $U \xrightarrow{sm} \mathfrak{X}$

We note $\Delta_{\mathfrak{X}}$ is representable implies for all algebraic spaces X, the maps $X \to \mathfrak{X}$ are representable. So, $U \to \mathfrak{X}$ being smooth surjection is well-defined.

Definition 4.2.4

A ***morphism of stacks*** over S, say $\mathfrak{X} \to \mathfrak{Y}$, is defined as an element of $\mathrm{Hom}_S(\mathfrak{X}, \mathfrak{Y})$, i.e. they are morphisms of fibered categories over S.

Lemma 4.2.5

Let $\mathfrak{X}$ be stack over S. Then $\Delta_{\mathfrak{X}}$ is representable iff for all S-schemes U, V, $u \in \mathfrak{X}(U)$ and $v \in \mathfrak{X}(V)$ with

$$\begin{tikzcd} U \times V \arrow[r, "p_U"] \arrow[dr, "p_V"] & U \arrow[d] \\ & V \end{tikzcd}$$

we have $\mathrm{Isom}(p_U^* u, p_V^* v)$ is an algebraic space. This is also equivalent to: for all $u, v \in \mathfrak{X}(U)$, $\mathrm{Isom}(u, v)$ is an algebraic space.

Proof. Equivalence of last 2 conditions: for all $u, v \in \mathfrak{X}(U)$,

$$\begin{tikzcd} \mathrm{Isom}(u,u) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Isom}(p_U^* u, p_U^* v) \arrow[d] \\ U \arrow[r, "\Delta"] & U \times U \end{tikzcd}$$

so we see $\mathrm{Isom}(p_U^* u, p_U^* v)$ is algebraic space imply $\mathrm{Isom}(u, v)$ is algebraic space. Conversely, $\mathrm{Isom}(p_U^* u, p_V^* v)$ is the special case with $U' = U \times V$, $u' = p_U^* u$, $v' = p_V^* v$.

Next, consider diagram

$$\begin{tikzcd} \mathfrak{Y} \arrow[r] \arrow[d, "{(u,v)}"] \arrow[dr, phantom, "\square"] & U \times V \arrow[d] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

Just like the proof for algebraic space, we see we get

$$\begin{tikzcd} \mathfrak{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d, "v"] \\ U \arrow[r, "u"] & \mathfrak{X} \end{tikzcd}$$

So, $\mathfrak{Y}(T)$ gives

$$\begin{tikzcd} T \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] \arrow[dr, phantom, "\overset{\xi}{\Nearrow}"] & & \\ & \mathfrak{X} & V \arrow[l, "v"] \arrow[d, "\pi_1"] \\ & U \arrow[u, "u"] \arrow[r, "\pi_2"] & S \end{tikzcd}$$

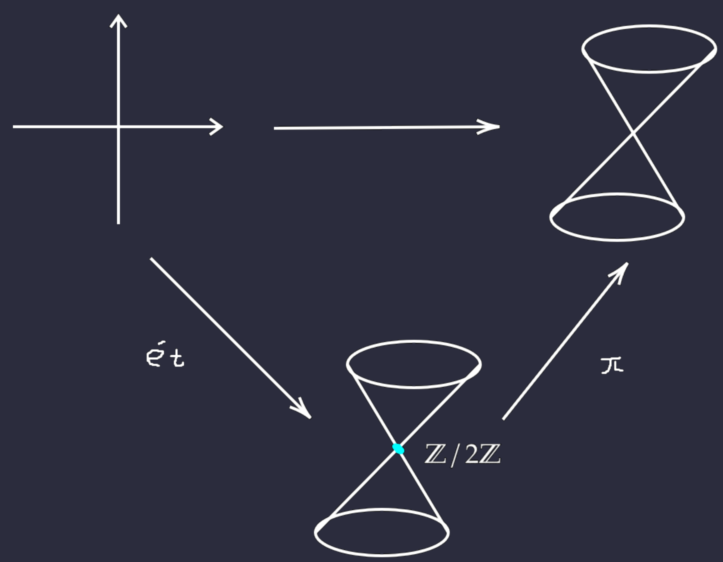
where $\xi : f(u) \xrightarrow{\sim} g(v)$. Then $\pi_1 f = \pi_g$ implies $T \xrightarrow{h} U \times_S V$ and hence $h^* p_U^* u = f^*(u) \underset{\xi}{\xrightarrow{\sim}} g^* v = h^* p_V^* v$. Thus we see $\xi \in \mathrm{Isom}(p_U^* u, p_V^* v)(T)$ which concludes the proof.

The next goal is to define one of the most important example of stacks, namely quotient stacks.

Example 4.2.6

Consider $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{A}^2$, i.e. $\mathbb{Z}/2\mathbb{Z} \circlearrowleft \mathbb{A}^2$ via $(x, y) \mapsto (-x, -y)$. In particular, we see $\mathbb{A}^2/(\mathbb{Z}/2\mathbb{Z}) := \operatorname{Spec} k[x, y]^{\mathbb{Z}/2\mathbb{Z}}$ where the ring $k[x, y]^{\mathbb{Z}/2\mathbb{Z}} = \{f(x, y) : f(x, y) = f(-x, -y)\} = k[x^2, xy, y^2]$ is the ring of invariant of the $\mathbb{Z}/2\mathbb{Z}$ action. We also note $k[x^2, xy, y^2] = k[a, b, c]/(ac - b^2)$ which is given by $\mathbb{A}^2 \to \mathbb{A}^2/(\mathbb{Z}/2\mathbb{Z})$. Thus we get $\mathbb{A}^2 \to \mathbb{A}^2/(\mathbb{Z}/2\mathbb{Z})$ is sort of like the xy-plane map to the cone defined by $ac - b^2$. But this is bad, because $V(ac - b^2)$ is singular.

That's why we want stakcs, where we replace the origin by the point $\mathbb{Z}/2\mathbb{Z}$, i.e. we get a "quotient stack" $[\mathbb{A}^2/(\mathbb{Z}/2\mathbb{Z})] =: \mathfrak{X}$, and we get the following diagram



Here the map π is whats called a coarse space map, and it is proper. In particular, π is isomorphism over X^{sm}. This is sort of like blow-up, but we didn't introduce any divisors, hence it is not a blow-up. This π is a "stacky resolution", and it is an example of Vistoli's canonical stacks.

In particular, we get a diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[dr, "\text{can stack}"'] & \tilde{X} \arrow[d, "\text{minimal res}"] \\ & X \end{tikzcd}$$

where $\tilde{X}$ is the minimal resolution of X, and $\mathfrak{X}$ is the canonical stack we discussed above, and X is any surface with mild singularity.

In physics, we get McKay correspondence that comparing $\mathfrak{X}$ and $\tilde{X}$. We see a lot of interesting math about comparing the two, and it also relates to deriving categories.

Before we jump to definition, we give one or two words about the idea. Say we have group scheme $G \circlearrowleft X$ over S. Then we get $X \to [X/G]$ and what we want is to have the arrow $X \to [X/G]$ to be a G-torsor.

We don't realy know what $X \to [X/G]$ is G-torsor means, thus we want to pullback and get

$$\begin{tikzcd} G \circlearrowleft P \arrow[r, "\text{G-equiv}"] \arrow[d, "\text{G-torsor}"'] \arrow[dr, phantom, "\square"] & G \circlearrowleft X \arrow[d, "\text{G-torsor}"] \\ T \arrow[r] & {[X/G]} \end{tikzcd}$$

This is going to be our definition.

Definition 4.2.7

Continue the above set-up, for any scheme T the category $[X/G](T)$ is defined as follows. The objects are

$$\begin{tikzcd} P \arrow[r, "\text{G-equiv}", "\pi"'] \arrow[d, "\text{G-tors}"'] & X_T = X \times_S T \arrow[dl] \\ T & \end{tikzcd}$$

The morphisms $(T', P', \pi') \to (T, P, \pi)$ is given by a pair $(f, f^\flat)$, where $f : T' \to T$ is S-morphism of schemes, $f^\flat : P' \to f^*P$ is an isomorphism of $G_{T'}$-torsors on $((\mathbf{Sch})/T')$ such that the induced diagram

$$\begin{tikzcd} P' \arrow[r, "f^\flat"] \arrow[dr, "\pi'"'] & f^*P \arrow[d, "f^*\pi"] \\ & X \times_S T' \end{tikzcd}$$

commutes.

We let $\mathfrak{X} = [X/G]$. Why is $\mathfrak{X}$ a stack? We know G-torsors are sheaves, so we have descent for sheaves. Then descent as sheaf with G-action we see the G-action $G \times P \to P$ is map of sheaves, so those descent as well. We see $G \circlearrowleft P$ is torsor if $G \times P \xrightarrow{\sim} P \times P$ given by $(g, p) \mapsto (p, gp)$ and we can check this isomorphism locally.

Why is $\Delta_{\mathfrak{X}}$ representable? Let (P_1, π_1), (P_2, π_2) be over T. Let

$$I = \mathrm{Isom}((P_1, \pi_1), (P_2, \pi_2))$$

We want to show I is algebraic space.

First, we claim (its an exercise!) that if $\mathcal{F} \to W$ where W is scheme and $\mathcal{F}$ a sheaf, then we can check $\mathcal{F}$ is algebraic space étale locally on W.

Thus, to check I is algebraic space, we can make étale base change on T so that (P_i, π_i) are trivial torsors. Thus now we have

$$\begin{tikzcd} P_1 = G \arrow[r, "\xi", "\sim"'] \arrow[dr, "\pi_1"'] & P_2 = G \arrow[d, "\pi_2"] \\ & X \end{tikzcd}$$

where $\xi \in I$. However, note if we have $\xi(1) = g$, then for any h we get $\xi(h) = \xi(h \cdot 1) = h \cdot \xi(1) = hg$ and hence ξ is right multiplication by g. Thus we see $\pi_1(1) = \pi_2(g)$. Thus we see I has a very simple description:

$$\begin{tikzcd} I \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G_T \arrow[d, "{(1,g)}"] \\ X_T \arrow[r, "\Delta"] & X_T \times_T X_T \end{tikzcd}$$

In particular, since $G_T, X_T, X_T \times_T X_T$ are all schemes, hence I is a scheme, as desired.

Why is $\mathfrak{X}$ has smooth cover by scheme?

We see we get $X \xrightarrow[sm]{q} [X/G] = \mathfrak{X}$ where q is defined by the diagram

$$\begin{tikzcd} U := G \times X \arrow[r, "\text{action}"] \arrow[d, "\text{G-tors}"] & X \\ X & \end{tikzcd}$$

Why is q a smooth surjection? Well, we get

$$\begin{tikzcd} I := \mathrm{Isom}((U,\sigma),(P,\pi)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "q(u,\sigma)"] \\ T \arrow[r, "(P,\pi)"] & \mathfrak{X} \end{tikzcd}$$

where $G = U \to P$ is given by $1 \mapsto p$. So, we get map $I \xrightarrow{\sim} P$ given by $(U \xrightarrow{f} P) \mapsto f(1)$.

The point is that we get

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, two heads, "sm"] \arrow[dr, phantom, "\square"] & X \arrow[d, "q"] \\ T \arrow[r, "(P,\pi)"] & \mathfrak{X} \end{tikzcd}$$

and since $P \to T$ is smooth, q is smooth surjection as desired. Thus, we showed, every torsor is the pullback of the torsor $X \to [X/G]$!

Example 4.2.8

Let G be S-smooth group scheme, and $G \circlearrowleft S$ be the trivial action. Then we define classifying stack of G to be $BG := [S/G]$. Hence we get

$$\begin{tikzcd} P \arrow[r, "\text{G-equiv}"] \arrow[d, "\text{G-tors}"] & S \arrow[d] \\ T \arrow[r] & BG \end{tikzcd}$$

However, note the action is trivial, thus G-equivariant $P \to S$ is just any arrow $P \to S$. Hence we see $(BG)(T)$ is just G-torsors on T.

Example 4.2.9

Let $G = \mathbb{G}_m = \mathrm{GL}_1$, then $B\mathbb{G}_m$ is just line bundles, and $B\,\mathrm{GL}_n$ is vector bundles.

Definition 4.2.10

We say $\mathfrak{X}$ is quotient stack if $\mathfrak{X} \cong [X/G]$ for some X, G.

In the above, we talked about quotient stacks $\mathfrak{X} = [X/G]$ where G is a group scheme over S and $G \circlearrowleft X$ with X a S-scheme.

We also showed that $X \to [X/G]$ is universal, in the sense that if we have $T \to [X/G]$ then we get the following diagram

$$\begin{tikzcd} P \arrow[r, "\text{G-equiv}"] \arrow[d, "\text{G-tors}"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "sm"', "\text{G-tors}"] \\ T \arrow[r] & {[X/G]} \end{tikzcd}$$

We also talked about examples. In particular, $\mathbb{P}^n = [\mathbb{A}^{n+1} \backslash 0/\mathbb{G}_m]$ and more generally, if $G \circlearrowleft X$ is a free action, i.e. all stablizers are trivial, then $[X/G]$ is an algebraic space that is exactly X/G. For example, if $X = \operatorname{Spec} A$ then $[X/G] = X/G = \operatorname{Spec}(A^G)$.

Proposition 4.2.11

If $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ are Artin stacks with

$$\begin{tikzcd} & \mathfrak{Y} \arrow[d] \\ \mathfrak{X} \arrow[r] & \mathfrak{Z} \end{tikzcd}$$

then $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is an Artin stack.

Let $\mathfrak{X}$ be an algebraic stack and let $x : \operatorname{Spec} R \to \mathfrak{X}$ be an R-valued point, then we define $G_x = \operatorname{Aut}_{\mathfrak{X}(R)}(x) = \operatorname{Aut}(x) = \operatorname{Isom}_{\mathfrak{X}}(x, x)$ as the fiber product

$$\begin{tikzcd} G_x := \operatorname{Aut}_{\mathfrak{X}(R)}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} R \arrow[d, "{(x,x)}"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

We see this is representable by an algebraic space over $\operatorname{Spec} R$ (by Lemma 4.2.5), and as a matter of fact, it is a group algebraic space. In fact, one can show if the diagonal of $\mathfrak{X}$ is quasi-separated (not defined yet, but you can guess what it means), then G_x is a group scheme locally of finite type.

Remark 4.2.12

If $\mathfrak{X}, \mathfrak{X}' \to \mathfrak{Y}$ and $z \in (\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}')(\operatorname{Spec} R)$, then $G_z = G_x \times_{G_y} G_{x'}$, where x, x', y are the images of x' in $\mathfrak{X}, \mathfrak{X}'$ and $\mathfrak{Y}$, respectively.

By Proposition 2.2.15, we see G_x can be identified with the Cartesian product

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} R \arrow[d] \\ \mathfrak{X} \arrow[r] & \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} \end{tikzcd}$$

and hence it is reasonable to define a stack that packs all the stablizer information using this diagram. This leads to the following definition.

Definition 4.2.13

For $\mathfrak{X}$, we define the ***inertia stack*** $I_{\mathfrak{X}} = I\mathfrak{X}$ to be the pullback

$$\begin{tikzcd} I_{\mathfrak{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "\Delta"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

Suppose we have $h \in \mathfrak{X}(T)$ with T a scheme, then $I\mathfrak{X}(h) = \mathrm{Isom}_T(h,h)$. If $\alpha : h \to h'$ is a morphism over $T \to T'$, then the natural pullback functor $\alpha^* : \mathrm{Aut}_{T'}(h') \to \mathrm{Aut}_T(h)$ is defined as follows: for $\beta \in \mathrm{Aut}_{T'}(a')$, the image $\alpha^*(\beta)$ is the unique dotted arrow making the diagram

$$\begin{tikzcd} h \arrow[r, dashed, "\alpha^*(\beta)"] \arrow[rr, bend right, "\beta\circ\alpha"'] & h \arrow[r, "\alpha"] & h' \end{tikzcd}$$

commute. If $\alpha : h \to h$ is an isomorphism over the identity, then $\alpha^*\beta = \alpha^{-1} \circ \beta \circ \alpha$

Example 4.2.14

Let $G \to S$ be a group scheme acting on S-scheme U, and $\mathfrak{X} = [U/G]$ be the quotient stack. Then there is a Cartesian diagram

$$\begin{tikzcd} S_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ I\mathscr{X} \arrow[r] & \mathscr{X} \end{tikzcd}$$

where S_U is the stabilizer group scheme, i.e. the fibered product of the action map $G \times U \to U \times U$ and the diagonal $U \to U \times U$.

Example 4.2.15

The inertia class of the classifying stack $B\mathbb{G}_m$ is $I_{B\mathbb{G}_m} \cong \mathbb{G}_m \times B\mathbb{G}_m$. Similarly, if we let $\mathbb{G}_m$ acts on $\mathbb{G}_m \times \mathbb{A}^1$ via the trivial action times the scaling action, and we let $V(x(t-1))$ be the $\mathbb{G}_m$-invariant closed subscheme, then $I_{[\mathbb{A}^1/\mathbb{G}_m]} \cong [V(x(t-1))/\mathbb{G}_m]$.

Example 4.2.16

More generally, let G be a finite group acting on scheme U, and $\mathfrak{X} = [U/G]$. Then the inertia stack $I_{\mathfrak{X}}$ is isomorphic to

$$\coprod_{g\in G} [U^g/C(g)]$$

where $U^g = \{x \in U : gx = x\}$ and $C(g) = \{s \in G : gs = sg\}$ the centralizer of

G. Alternatively, this is the fiber product of the diagonal $U \to U \times U$ and the map $U \to U \times U$ defined by $x \mapsto (x, gx)$.

4.3 Properties of Algebraic Stacks

The next topic is properties for stacks and morphisms.

Definition 4.3.1

Let P be a property that is local for smooth topology. Then we say $\mathfrak{X}$ has P if there exists smooth cover $X \twoheadrightarrow \mathfrak{X}$ with X scheme, such that X has P.

Example 4.3.2

P could be locally Noetherian, regular, of finite type over S, of finite presentation over S.

Lemma 4.3.3

If P is local for smooth topology, and $\mathfrak{X}$ has P, and for Y a scheme we have $Y \xrightarrow{sm} \mathfrak{X}$ then Y has P.

Proof. Consider

$$\begin{tikzcd} Z \arrow[r, "sm"] \arrow[d, "sm"] \arrow[dr, phantom, "\square"] & X \arrow[d, "sm"] \\ Y \arrow[r, "sm"] & \mathfrak{X} \end{tikzcd}$$

where we assume X has P. But X has P implies Z has P and hence Y has P as desired.

Remark 4.3.4

The proof shows that if $Y \to \mathfrak{X}$ is a morphism, then smooth locally on Y, $Y \to \mathfrak{X}$ factors through $X \to \mathfrak{X}$.

Definition 4.3.5

If $f : \mathfrak{X} \to \mathfrak{Y}$ is morphism of Artin stacks, then a ***chart for*** f is a diagram

$$\begin{tikzcd} X \arrow[r, two heads, "sm"] \arrow[dr] \arrow[rr, bend left, "g"] & \mathfrak{Z} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & Y \arrow[d, two heads, "sm"] \\ & \mathfrak{X} \arrow[r, "f"'] & \mathfrak{Y} \end{tikzcd}$$

where X,Y are algebraic spaces. If P is property of morphism of schemes stable under base change, local on source and target for smooth topology, then we say $f:\mathfrak{X}\to\mathfrak{Y}$ ***has*** P if g has P. In this case we also say that this ***chart*** g ***has*** P.

Example 4.3.6

P could be smooth, locally of finite presentation, surjective, etc.

Example 4.3.7

If we are given quotient stack $[X/G]$ over S, and

$$\begin{tikzcd} X \arrow[r, two heads, "sm"] \arrow[rr, bend right, "g"'] & \mathfrak{X} \arrow[r, "f"] & S \end{tikzcd}$$

Then $\mathfrak{X}/S$ is smooth iff X/S is smooth. For example, we see $[\mathbb{A}^2/(\mathbb{Z}/2)]$ is smooth because $\mathbb{A}^2$ is smooth. On the other hand, $\mathbb{A}^2/(\mathbb{Z}/2)$ is singular as its equal $\operatorname{Spec} k[x,y]^{\mathbb{Z}/2}$.

Proposition 4.3.8

The morphism $f:\mathfrak{X}\to\mathfrak{Y}$ has P iff every chart has P.

Proof. We start with a chart

$$\begin{tikzcd} X \arrow[r, two heads] & \mathfrak{Z} \arrow[r] \arrow[d] & Y \arrow[d] \\ & \mathfrak{X} \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

Then we get another chart

$$\begin{tikzcd} X \arrow[rr] & & \mathfrak{Z} \arrow[rr] \arrow[dd] & & Y \arrow[dd] & \\ & X' \arrow[rr] & & \mathfrak{Z}' \arrow[rr] \arrow[dl] & & Y' \arrow[dl] \\ & & \mathfrak{X} \arrow[rr] & & \mathfrak{Y} & \end{tikzcd}$$

Now we want that: $X\to Y$ has P iff $X'\to Y'$ has P.

Now take pullbacks of the squares of the two sides, we get

$$\begin{tikzcd}
& X'' \arrow[rr] && \mathfrak{Z}'' \arrow[dl, two heads] \arrow[dd] && Y'' \arrow[dl, two heads] \arrow[dd] \\
X \arrow[rr] && \mathfrak{Z} \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\square"] && Y \arrow[dd] \arrow[dr, phantom, "\square"] \\
& X' \arrow[rr] && \mathfrak{Z}' \arrow[dl] \arrow[rr] && Y' \arrow[dl] \\
&& \mathfrak{X} \arrow[rr] && \mathfrak{Y}
\end{tikzcd}$$

and we also get natural arrows from $\mathfrak{Z}'' \to Y''$. Viz we get

$$\begin{tikzcd}
& X'' \arrow[rr] && \mathfrak{Z}'' \arrow[dl, two heads] \arrow[dd] \arrow[rr] && Y'' \arrow[dl, two heads] \arrow[dd] \\
X \arrow[rr] && \mathfrak{Z} \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\square"] && Y \arrow[dd] \arrow[dr, phantom, "\square"] \\
& X' \arrow[rr] && \mathfrak{Z}' \arrow[dl] \arrow[rr] && Y' \arrow[dl] \\
&& \mathfrak{X} \arrow[rr] && \mathfrak{Y}
\end{tikzcd}$$

Thus it is good enough to show $X \to Y$ has P iff $X'' \to Y''$ has P, i.e. it is good enough to handle the case when $Y' \to \mathfrak{Y}$ factors as $Y' \twoheadrightarrow Y \twoheadrightarrow \mathfrak{Y}$.

First, consider the case $X' = X \times_{\mathfrak{Z}} \mathfrak{Z}'$ and we get diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[rr, bend left, "g'"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \mathfrak{Z}' \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & Y' \arrow[d, two heads] \\
X \arrow[r] \arrow[rr, bend left, "g"] & \mathfrak{Z} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & Y \arrow[d, two heads] \\
& \mathfrak{X} \arrow[r] & \mathfrak{Y}
\end{tikzcd}$$

where g' is pullback of g under $Y' \twoheadrightarrow Y$. We see P is smooth local on the target so g has P iff g' has P.

Now, we just need to compare two charts with the same Y. To see this, we note we have the following diagram

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d, two heads, "\pi"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, two heads] \arrow[dr, dashed, "g'"] & \\
X \arrow[r, two heads] \arrow[rr, bend right, "g"'] & \mathfrak{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\
& \mathfrak{X} \arrow[r] & \mathfrak{Y}
\end{tikzcd}$$

and we see g' has P iff $g'\pi' = g\pi$ has P iff g has P. This concludes the proof.

Next, we consider separatedness.

Proposition 4.3.9

Consider the diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "f"] \arrow[rr, bend right, "h"'] & \mathfrak{Y} \arrow[r, "g"] & \mathfrak{Z} \end{tikzcd}$$

of stacks with g representable. Then h is representable iff f is representable.

Proof. If f is representable, h is representable by definition. If h is representable, given $Z \xrightarrow{\alpha} \mathfrak{Y}$ we want $Z \times_{\mathfrak{Y}} \mathfrak{X}$ is an algebraic space. Thus we get the following diagram

$$\begin{tikzcd} \mathfrak{X} & X \arrow[l] \arrow[d] \\ \mathfrak{Y} \arrow[d, "g"'] \arrow[from=u, "f"'] \arrow[ur, phantom, "\square"] & Y \arrow[l] \arrow[d] \\ Z & Z \arrow[l] \arrow[ul, "\alpha"] \end{tikzcd}$$

where the bottom square is also Cartesian. We note X, Y are algebraic spaces because h, g are representable, respectively. Then, note we get a section $\beta : Z \to Y$ defined by α and hence we obtain the diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[d, "f"'] & X \arrow[l] \arrow[d] \\ \mathfrak{Y} \arrow[d, "g"'] & Y \arrow[l] \arrow[d, shift left] \\ Z & Z \arrow[l] \arrow[ul, "\alpha"] \arrow[u, shift left, "\beta"'] \end{tikzcd}$$

Now we see $\mathfrak{X} \times_{\mathfrak{Y},\alpha} Z = X \times_{Y,\beta} Z$ is an algebraic space as X, Y, Z are all algebraic spaces.

Proposition 4.3.10

Let $\mathfrak{X}, \mathfrak{Y}$ be Artin stacks over S and $f : \mathfrak{X} \to \mathfrak{Y}$, then $\Delta_{\mathfrak{X}/\mathfrak{Y}}$ is representable.

Proof. We have the following diagram (magic square)

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\Delta_{\mathfrak{X}/\mathfrak{Y}}"'] \arrow[rr, bend left, "\Delta_{\mathfrak{X}/S}", "repable"'] \arrow[dr, "f"'] & \mathfrak{X}\times_{\mathfrak{Y}}\mathfrak{X} \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X}\times_S\mathfrak{X} \arrow[d, "f\times f"] \\
 & \mathfrak{Y} \arrow[r, "repable", "\Delta_{\mathfrak{Y}/S}"'] & \mathfrak{Y}\times_S\mathfrak{Y}
\end{tikzcd}$$

Then $\Delta_{\mathfrak{Y}}$ representable implies g is representable. Hence $\Delta_{\mathfrak{X}}$ representable implies, by the proposition above, that $\Delta_{\mathfrak{X}/\mathfrak{Y}}$ is representable as desired.

Definition 4.3.11

We say $f : \mathfrak{X} \to \mathfrak{Y}$ is ***separated*** if $\Delta_{\mathfrak{X}/\mathfrak{Y}}$ is proper.

For stacks, Δ keeps track of Isom or Aut which is a group so that is is rarely a closed immersion. Hence this notion of separatedness is different from the scheme version, where we require Δ to be closed immersion. But in the scheme case this is in fact the same.

For X, Y schemes, $\Delta_{X/Y}$ is always an immersion, so they are separated and of finite type. Thus $\Delta_{X/Y}$ is proper iff $\Delta_{X/Y}$ universally closed but base changes of immersion is immersion and hence Δ is universally closed iff its closed.

Thus, $X \to Y$ separated in the usual definition ($\Delta_{X/Y}$ closed) iff its separated as stacks.

Remark 4.3.12

In general, conditions on the diagonal translates to conditions on the Isom sheaves, since the base change of $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ by a morphism $(a, b) : S \to \mathfrak{X} \times \mathfrak{X}$ from a scheme S is identified with the sheaf $\mathrm{Isom}(a, b)$, which is an algebraic space. In particular, we see $\mathfrak{X}$ has affine diagonal if and only if every algebraic space $\mathrm{Isom}_{\mathfrak{X}(S)}(a, b)$ is a scheme affine over S.

Example 4.3.13

Let $\mathfrak{X} = [X/G]$, then we see we get

$$\begin{tikzcd}
X \times_{\mathfrak{X}} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \times X \arrow[d] \\
\mathfrak{X} \arrow[r, "\Delta_{\mathfrak{X}}"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

where $X \times_{\mathfrak{X}} X$ is the Aut of universal torsor. Hence we see

$$\begin{tikzcd} X \times_{\mathfrak{X}} X \cong G \times X \arrow[r] \arrow[d, shift left] \arrow[dr, phantom, "\square"] & X \arrow[d, two heads] \\ X \arrow[u, shift left] \arrow[r, two heads] & \mathfrak{X} \end{tikzcd}$$

where $G \times X \cong X \times_{\mathfrak{X}} X$ because we has a section $X \to X \times_{\mathfrak{X}} X$. Thus, we see the above diagram's arrows are given by

$$\begin{tikzcd} (g,x) \arrow[rr, mapsto] \arrow[dd, mapsto] & & gx \\ & G \times X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, two heads] \\ x & X \arrow[r, two heads] & \mathfrak{X} \end{tikzcd}$$

So, we see

$$(g,x) \longmapsto (x, gx)$$

$$\begin{tikzcd} G \times X \arrow[r, "\Gamma"] \arrow[d] \arrow[dr, phantom, "\square"] & X \times X \arrow[d, two heads] \\ \mathfrak{X} \arrow[r, "\Delta"'] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

where Γ is the graph of the action. Therefore, we see $\mathfrak{X}$ is separated iff Γ is proper. By definition of $G \circlearrowleft X$ is called proper if Γ is proper.

Example 4.3.14

If X is separated and G is proper (e.g. G is finite) then

$$\begin{tikzcd} & G \times_S X \arrow[r] \arrow[dl] \arrow[dr, "proper"] & X \times_S X \arrow[r] \arrow[d, "sep"] \arrow[dr, phantom, "\square"] & X \arrow[d, "sep"] \\ G \arrow[dr, "proper"'] \arrow[rr, phantom, "\square"] & & X \arrow[r] \arrow[dl] & S \\ & S & & \end{tikzcd}$$

and hence Γ is proper, i.e. if X is separated, G proper (e.g. finite), then $[X/G]$ is separated.

Example 4.3.15

If A is Abelian variety, then BA is separated.

Next, we talk about the topological space $|\mathfrak{X}|$ associated with algebraic stack $\mathfrak{X}$.

This is defined by the set $|\mathfrak{X}|$ consisting of field-valued morphisms $x : \operatorname{Spec} K \to \mathfrak{X}$, and two points $x : \operatorname{Spec} K \to \mathfrak{X}$ and $x' : \operatorname{Spec} K' \to \mathfrak{X}$ are the same if there is a common field extension K'' of both fields, i.e. K''/K and K''/K' at the same time, and $x|_{\operatorname{Spec} K''} \cong x'|_{\operatorname{Spec} K''}$ in the category $\mathfrak{X}(K'')$. The topology on $|\mathfrak{X}|$ is given by the condition $U \subseteq |\mathfrak{X}|$ is open if and only if $U = |\mathfrak{U}|$ for some open substack $\mathfrak{U} \subseteq \mathfrak{X}$.

Example 4.3.16

The topological space of the quotient stack $|[\mathbb{A}^1_k/\mathbb{G}_m]|$ with the standard scaling action consists of two points with representatives $x_0 : \operatorname{Spec} k \xrightarrow{0} \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ and $x_1 : \operatorname{Spec} k \xrightarrow{1} \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$. In particular, the inclusion of the generic point $\operatorname{Spec} \kappa(x) \to \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to x_1.

While the stablizer group $G_{\overline{x}}$ depends on the choice of representatives $\overline{x} : \operatorname{Spec} k \to \mathfrak{X}$, its dimension, denoted by $\dim G_x$, is independent of this choice. Similarly, the properties of being smooth, unramified, affine, fintie and reduced are also independent of this choice. Next, we recover dimension and tangent spaces of algebraic stacks.

For dimension, we begin recall dim for a scheme, then algebraic spaces, and then algebraic stacks. If X is a scheme, then $\dim(X)$ is the Krull dimension of the topological space underlying X. Then, $\dim_x(X)$ is the minimum of the dimension of open subsets U containing X. If X is locally Noetherian, $\dim_x(X)$ coincides with the sup of the dimensions at x of the irreducible components of X passing through x.

Definition 4.3.17

If X is an algebraic space and $x \in |X|$, then

$$\dim_x(X) = \dim_u U$$

where U is any étale presentation $U \twoheadrightarrow X$, and $u \in U$ is any point lying over x. In particular,

$$\dim X = \sup_{x \in |X|} \dim_x(X)$$

In general, the dimension of the algebraic space X at a point x may not coincide with the dimension of the underlying space $|X|$ at x. For example, take $X = \mathbb{A}^1_k/\mathbb{Z}$, then X has dimension 1 at each of its points, but $|X|$ has indiscrete topology, and hence of Krull dimension 0.

However, if X is, for example, quasi-separated, then the dimension of X at x equal the dimension of $|X|$ at x.

Next, to define the dimension of an algebraic stack, we need the notion of relative dimension, at a point in the source, of a morphism whose source is an algebraic space, and whose target is algebraic stacks. We need this complication because points of an algebraic stack are not describable as morphisms from the spectrum of a field, but only as equivalence classes of such.

Definition 4.3.18

Let $f : T \to \mathfrak{X}$ is a locally of finite type morphism from an algebraic space to an algebraic stack, and if $t \in |T|$ is a point with image $x \in |\mathfrak{X}|$, then we define the ***relative dimension*** of f at t, denoted $\dim_t(T_x) = \dim_{f,t}(T_x)$, as follows: choose a morphism $\operatorname{Spec} k \to \mathfrak{X}$ that represents x, and choose a point $t' \in |T \times_{\mathfrak{X}} \operatorname{Spec} k|$ mapping to t under the projection to $|T|$, then

$$\dim_t(T_x) = \dim_{t'}(T \times_{\mathfrak{X}} \operatorname{Spec} k)$$

Note in the above definition, since T is an algebraic space, $\mathfrak{X}$ an algebraic stack, the fiber product $T \times_{\mathfrak{X}} \operatorname{Spec} k$ is an algebraic space, and thus the dimension of a point make sense, by taking an étale presentation.

Next, to define dimension of a locally Noetherian algebraic stacks, we recall the following lemma.

Lemma 4.3.19

If $f : U \to X$ is a smooth morphism of locally Noetherian algebraic spaces, and $u \in |U|$ with image $x \in |X|$, then

$$\dim_u(U) = \dim_x(X) + \dim_{f,u}(U_x)$$

where $\dim_{f,u}(U_x)$ is from Definition 4.3.18.

The proof can be found in Stack project, 0AFI.

Definition 4.3.20

Let $\mathfrak{X}$ be locally Noetherian algebraic stacks, and $x \in |\mathfrak{X}|$, then we define $\dim_x(\mathfrak{X})$ as follows. Let $f : U \to \mathfrak{X}$ be a smooth morphism from a scheme to $\mathfrak{X}$ containing x in its image, u be any point of $|U|$ mapping to x, and define

$$\dim_x(\mathfrak{X}) = \dim_u(U) - \dim_{f,u}(U_x)$$

In particular, $\dim \mathfrak{X} = \sup_{x \in |\mathfrak{X}|} \dim_x(\mathfrak{X})$.

This definition is justified by the formula of the above lemma, and we can use this to verify $\dim_x(\mathfrak{X})$ is well-defined, independent of the choices used to compute it.

Example 4.3.21

If U is a scheme of pure dimension with an action of an affine algebraic group G over field k, then

$$\dim[U/G] = \dim U - \dim G$$

Next, the definition of tangent space is defined similarly to the one for schemes.

Definition 4.3.22

If $\mathfrak{X}$ is an algebraic stack and $x : \operatorname{Spec} k \to \mathfrak{X}$, we define the ***Zariski tangent space*** $T_{\mathfrak{X},x}$ as the set of diagrams

$$\begin{tikzcd} \operatorname{Spec} k \arrow[d] \arrow[dr, "x"] & \\ \operatorname{Spec} k[\epsilon] \arrow[r, "\tau"'] & \mathfrak{X} \end{tikzcd}$$

mod out the equivalence relation $(\tau, \alpha) \sim (\tau', \alpha')$ if there is an isomorphism $\beta : \tau \xrightarrow{\sim} \tau'$ in $\mathfrak{X}(k[\epsilon])$ compatible with α and α', i.e. $\alpha' = \beta|_{\operatorname{Spec} k} \circ \alpha$.

Note in the above, the arrow α means we require $\alpha : x \xrightarrow{\sim} \tau|_k$.

Proposition 4.3.23

If $\mathfrak{X}$ is an algebraic stack with affine diagonal and $x \in \mathfrak{X}(k)$, then $T_{\mathfrak{X},x}$ is naturally a k-vector space.

We note $T_{\mathfrak{X},x}$ is naturally a representation of the group G_x, which given set-theoretically by $g \cdot (\tau, \alpha) = (\tau, g \circ \alpha)$ for $g \in G_x$ and $(\tau, \alpha) \in T_{\mathfrak{X},x}$.

Example 4.3.24

Consider a smooth connected and projective curve $[C] \in M_g(k)$ defined over k of genus $g \geq 2$. Then $T_{M_g,[C]} = H^1(C, T_C)$ by deformation theory. Since $\deg T_C < 0$, $H^0(C, T_C) = 0$ and Riemann-Roch implies

$$\dim T_{M_g,[C]} = \dim H^1(C, T_C) = -\chi(T_C) = -(\deg T_C + (1 - g)) = 3g - 3$$

To conclude this section, we introduce the notion of residual gerbes, which is the stack version of residue fields.

Recall that attached to every point $x \in X$ a scheme, we get residue field $\kappa(x)$ and a monomorphism $\operatorname{Spec} \kappa(x) \to X$ with image x. To define similar notion for stacks, we note the existence of non-trivial stablizers prevents field-valued points from being monomorphism.

Definition 4.3.25

Let $\mathfrak{X}$ be an algebraic stack and $x \in |\mathfrak{X}|$ a point. We say the ***residual gerbe at*** x ***exists*** if there is a reduced Noetherian algebraic stack $\mathfrak{G}_x$ and a monomorphism $\mathfrak{G}_x \to \mathfrak{X}$ such that $|\mathfrak{G}_x|$ is a point mapping to x. If it exists, we call $\mathfrak{G}_x$ the ***residual gerbe at*** x.

In later chapter we will see the residual gerbe is unique if it exists, and show $\mathfrak{G}_x$ is indeed a gerbe over a field $\kappa(x)$, the residual field of x.

Showing the existence of residual gerbes fairly straightforward in the case of a finite type point (i.e. meaning the point $x \in |\mathfrak{X}|$ has a representative $\operatorname{Spec} k \to \mathfrak{X}$ locally of finite type). However, more generally, this residual gerbes exist for any point of a quasi-separated algebraic stack.

Remark 4.3.26

If X is a Noetherian scheme, then $x \in X i$ is of finite type iff $x \in X$ is locally closed. More generally, $\operatorname{Spec} k \to X$ is of finite type if and only if the image $x \in X$ is locally closed and $\kappa(x)/k$ is a finite extension.

An example of a finite type point of a scheme that is not closed is the generic point of a DVR.

Example 4.3.27

Let $\mathfrak{X}$ be an algebraic stack, then $x \in |\mathfrak{X}|$ is of finite type if and only if there is a scheme U, a closed point $u \in U$, and a smooth morphism $(U, u) \to (\mathfrak{X}, x)$.

Proposition 4.3.28

If $\mathfrak{X}$ is Noetherian, $x \in \mathfrak{X}$ a finite type point, then the residual gerbe $\mathfrak{G}_x$ exists and is a regular algebraic stack, and the morphism $\mathfrak{G}_x \to \mathfrak{X}$ is locally closed immersion. In addition, if $\mathfrak{X}$ is of finite type over a field k and $x \in \mathfrak{X}(k)$ has an affine smooth stablizer, then $\mathfrak{G}_x = BG_x$.

Proof. We only prove the first statement. After replacing $\mathfrak{X}$ with $\overline{\{x\}}$, we can assume $\mathfrak{X}$ is reduced and $x \in |\mathfrak{X}|$ is dense. Let $\operatorname{Spec} k \to \mathfrak{X}$ be a finitely presented representative of x. By generic flatness, $\operatorname{Spec} k \to \mathfrak{X}$ is flat and thus its image, which is $x \in |\mathfrak{X}|$, is open. The corresponding open substack $\mathfrak{G}_x \subseteq \mathfrak{X}$ is the residual gerbe. Since $\operatorname{Spec} k \to \mathfrak{G}_x$ is fppf and the property of being regular descends under fppf topology, $\mathfrak{G}_x$ is regular.

4.4 Deligne-Mumford Stacks

Last time we talked about separatedness. In particular, if X is separated over S, G proper over S, then $[X/G]$ is separated over S.

The next topic is to work towards the first big theorem in stacks.

Definition 4.4.1

If $\mathfrak{X}/S$ is Artin stack, then we say it is ***Deligne-Mumford*** (DM) if there exists scheme X with $X \xrightarrow{\text{ét}} \mathfrak{X}$.

In general, its not easy to check when a stack is DM. To check this, we recall the notion of formally étale/smooth. That is, we say $X \xrightarrow{f} Y$ is formally étale/smooth if for all diagrams

$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] & Y \end{tikzcd}$$

with $I^2 = 0$ in A, there exists unique arrow/exists arrow from $\operatorname{Spec} A$ to X.

So, étale means exists unique arrow, smooth means exists arrow, and now we define formally unramified, means unique/at most one arrow.

Definition 4.4.2

We say $X \xrightarrow{f} Y$ is ***formally unramified*** if for all diagrams

$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, shift left, "\alpha"] \arrow[ur, shift right, "\beta"'] & Y \end{tikzcd}$$

with $I^2 = 0$ then $\alpha = \beta$.

Theorem 4.4.3

If $\mathfrak{X}$ is Artin stack over S, then $\mathfrak{X}$ is DM iff $\Delta_{\mathfrak{X}/S}$ is formally unramified.

We first state some corollaries, before we prove this.

Corollary 4.4.3.1

If $\mathfrak{X}$ is Artin stack over S, then $\mathfrak{X}$ is algebraic space over S iff for all $x \in \mathfrak{X}(U)$, $\operatorname{Aut}(x)$ is trivial, i.e. algebraic spaces are Artin stacks with no stablizers.

Proof. First, if $\mathfrak{X}$ is algebraic space, then $\mathfrak{X}$ is sheaf, i.e. fibered in sets, so no automorphisms.

Conversely, if

$$\begin{tikzcd} \operatorname{Isom}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "{(x,y)}"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times_S \mathfrak{X} \end{tikzcd}$$

If $\operatorname{Isom}(x,y) = \emptyset$, then it is a scheme and $\operatorname{Isom}(x,y) \to U$ is formally unramified. If $\operatorname{Isom}(x,y) \neq \emptyset$, then it is an $\operatorname{Aut}(x)$-torsor. That is, if we have two isomorphisms

$x \overset{\beta}{\underset{\alpha}{\rightrightarrows}} y$ then α, β are related by unique automorphism $\beta^{-1} \circ \alpha$. By assumption, $\mathrm{Aut}(x) \to U$ is the trivial group scheme, i.e. $U \xrightarrow{\mathrm{Id}} U$. So it is affine group scheme, and so $\mathrm{Isom}(x, y)$ is a scheme. Also, $\mathrm{Isom}(x, y) \to U$ is a monomorphism, so it is formally unramified. Therefore Δ is formally unramified and representable by schemes. Now by the big theorem above, we see there exists a étale cover.

Lastly, $\mathfrak{X}$ is a sheaf because the automorphisms are trivial and hence $\mathfrak{X}$ is algebraic space as desired.

Remark 4.4.4

Suppose $\Delta_{\mathfrak{X}}$ is locally of finite presentation. Then the theorem says $\mathfrak{X}$ is DM iff for all $k = \overline{k}$ and $\mathrm{Spec}\, k \xrightarrow{x} \mathfrak{X}$, the automorphism group $\mathrm{Aut}(x) \to \mathrm{Spec}\, k$ is a finite group.

Why is this? We know $\Delta_{\mathfrak{X}}$ is formally unramified iff $\mathrm{Isom}(x, y) \to U$ is formally unramified for all $x, y \in \mathfrak{X}(U)$. However, since $\Delta_{\mathfrak{X}}$ is locally of finite presentation, $\mathrm{Isom}(x, y) \to U$ is locally of finite presentation. Hence, we can check formally unramified for $\mathrm{Isom}(x, y) \to U$ on geometric points, i.e. for all $k = \overline{k}$ with $\mathrm{Spec}\, k \xrightarrow{f} U$ we can check $\mathrm{Isom}(f^*x, f^*y) \to \mathrm{Spec}\, k$ formally unramified.

If $\mathrm{Isom}(f^*x, f^*y) = \emptyset$ there is nothing to do, else (i.e. its non-empty) it is isomorphic to $\mathrm{Aut}(x)$. But then $\mathrm{Aut}(x) \to \mathrm{Spec}\, k$ is locally of finite presentation, so its formally unramified if and only if étale if and only if $\mathrm{Aut}(x)$ is finite if and only if $\mathrm{Aut}(x)$ is a group.

Example 4.4.5

From the above remark, we see $B\mathbb{G}_m$ is not DM as $\mathbb{G}_m$ is not finite. More generally, BG is not DM if G is a positive dimension group scheme.

Example 4.4.6

A long time ago we talked about moduli space of genus g curves M_g, where $M_g(T)$ are given by $C \xrightarrow{\pi} T$ with π smooth proper and on geometric fibers it is genus g curve.

When Deligne and Mumford defined DM stacks, they showed that for $g \geq 2$, M_g is DM and defined a compactification $\overline{M}_g$. We will go through the idea of how to show M_g is DM for $g \geq 2$.

We start with a curve C of genus $g \geq 2$ over a ACF $k = \overline{k}$. We want $\mathrm{Aut}(C) \to \mathrm{Spec}\, k$ to be formally unramified (i.e. we want $\mathrm{Aut}(C)$ is finite group). So we want to show for all diagrams

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} A' \arrow[r] \arrow[ur, shift left, "\alpha"] \arrow[ur, shift right, "\beta"'] & Y
\end{tikzcd}$$

with $A = A'/I$, we have $\alpha = \beta$. Now, we consider the diagram

$$\begin{tikzcd}
C_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_{A'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\
\operatorname{Spec} A \arrow[r] & \operatorname{Spec} A' \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

where we get two arrows from $C_{A'}$ to itself (α, β) and one arrow from C_A to itself (γ), where α, β both reduce to be γ over A. Now by deformation theory, "$\alpha - \beta$" is a class in $H^0(T_{C_A/A})$, where $T_{C_A/A}$ is tangent bundle. However, the degree of tangent bundles are given by $2-2g$. Hence we see for $g \geq 2$, the degree become negative, i.e. $H^0(T_{C_A/A}) = 0$ and hence $\alpha = \beta$, as desired.

What about $g = 0, 1$?

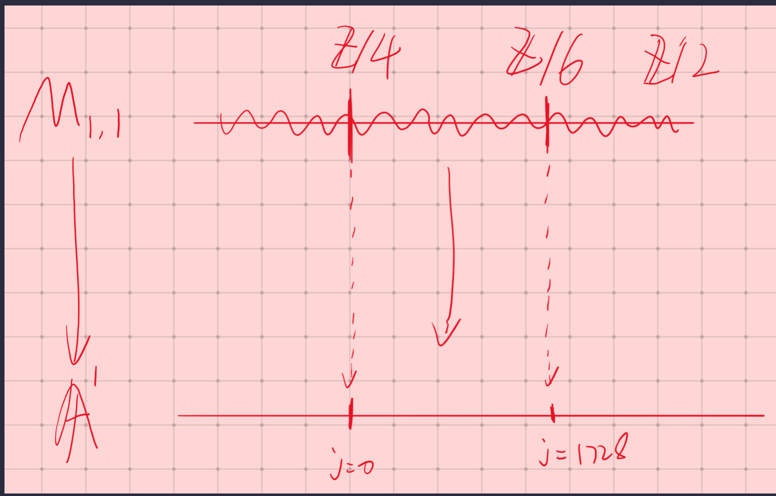
If $g = 0$, then M_0 is just $\mathbb{P}$ over ACF $k = \overline{k}$, and over T it is not. It is a Brauer-Severi variety. These are $\operatorname{Aut}(\mathbb{P}^1)$-torsors and so $M_0 = B(\mathrm{PGL}_2)$. We know $\dim \mathrm{PGL}_2 = 3 > 0$, so M_0 is Artin, not DM. In particular, the dimension of M_0 is dimension of a point subtract dimension of PGL_2, i.e. $\dim M_0 = -3$.

Next, we consider $M_{0,3}$, the moduli space of genus 0 curves with 3 marked points. Let C be a curve with $g = 0$ and three marked points, then its isomorphic to $\mathbb{P}^1$ with three additional points. Thus $M_{0,3}$ =points.

In terms of deformation theory, we get $\alpha - \beta$ lives in $H^0(T(-3\quad \mathrm{pt}s))$, where $T(-3\quad \mathrm{pt}s)$ is twisted down by three points, thus the degree is $2-2g-3 < 0$.

Similarly, M_1 is Artin, but $M_{1,1}$ is genus 1 curves with one marked point, which is just the moduli space of elliptic curves. In particular, $\dim M_{1,1} = 1$, which is exactly the j-invariant of elliptic curves. We get a map $M_{1,1} \to \mathbb{A}^1$ which sends elliptic curve E to isomorphism class of E (i.e. sends it to the j-invariant).

We give a rough picture of what this looks like:



generically we have $\mathbb{Z}/2$-stablizers since E has automorphisms equal multiply by -1. However, for $j=0$ and $j=1728$, we have more automorphisms: $\mathbb{Z}/4$ and $\mathbb{Z}/6$.

In particular, by using the Weierstrass form $y^2=x^3+ax+b$ for elliptic curves, we can show that if we invert the primes 2 and 3, then we have an isomorphism

$$M_{1,1}\times_{\mathbb{Z}}\mathbb{Z}[1/6]\cong[(\mathbb{A}^2\backslash V(\Delta))/\mathbb{G}_m]$$

where the action is given by $t\cdot(a,b)=(t^4a,t^6b)$, and Δ is the discriminant $4a^3+27b^2$.

We don't have time for the proof of the big theorme, but we go through the idea first.

If $X\to Y$ is a map of schemes, its smooth iff we can find Zariski cover $U\twoheadrightarrow X$ with

$$\begin{tikzcd}
U \arrow[rr, "et"] \arrow[dr] & & \mathbb{A}^n_Y \arrow[dd] \\
& X \arrow[dr, "f"'] & \\
& & Y
\end{tikzcd}$$

where $n=\dim X-\dim Y$ and F comes from the following: $\Omega^1_{X/Y}$ is locally free, we look locally on U where $U^1_{X/Y}|_U$ is free. We choose basis $df_1,\dots,df_n$ which yields map to $\mathbb{A}^n_Y$ coming from $(f_1,\dots,f_n)$.

For us, we have smooth $X\twoheadrightarrow\mathfrak{X}$, we would like to have something like "$\Omega^1_{X/\mathfrak{X}}$". We look étale locally where $\Omega^1_{X/\mathfrak{X}}$ is free to get

$$\begin{tikzcd}
X \arrow[r, "et"] \arrow[dr, "sm"] & \mathbb{A}^n_{\mathfrak{X}} \arrow[d] \\
& \mathfrak{X}
\end{tikzcd}$$

Formally unramified will allow us to "slice" $\mathbb{A}^n_{\mathfrak{X}}\to\mathfrak{X}$ to get $W\subseteq\mathbb{A}^n_{\mathfrak{X}}$ with étale arrow

$W \to X$

$$\begin{tikzcd} \mathbb{A}^n_{\mathfrak{X}} \arrow[r] & \mathfrak{X} \\ W \arrow[u, "\subseteq"] \arrow[ur, "et"] & \end{tikzcd}$$

Before we end, we talk about how to define $\Omega^1_{X/\mathfrak{X}}$.

We don't realy know what to do, hence the first thing is to descent. Thus, consider the following diagram

$$\begin{tikzcd} Z \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[r, two heads, "\pi"] \arrow[d, dashed, two heads, "\pi'"] \arrow[dr, phantom, "\square"] & X \arrow[d, two heads, "sm"] \\ Y \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r, two heads, "sm"'] & \mathfrak{X} \end{tikzcd}$$

We have $\Omega^1_{Y/X} = \Omega^1_{\pi'}$ and we get canonical isomorphism $p^*\Omega^1_{Y/X} \cong \Omega^1_{Z/Y} \cong q^*\Omega^1_{Y/X}$. Thus it satisfies the cocycle condition. Thus by descent of coherent sheaves, we get $\Omega^1_{X/\mathfrak{X}}$ such that $\pi^*\Omega^1_{X/\mathfrak{X}} \cong \Omega^1_{Y/X}$ where Y/X is via the map $\pi' : Y \to X$.

Moreover, $\Omega^1_{X/\mathfrak{X}}$ is locally free sheaf on X because $\pi^*\Omega^1_{X/\mathfrak{X}} \cong \Omega^1_{Y/X}$ is.

In addition, we get $\Omega^1_{X/S} \to \Omega^1_{X/\mathfrak{X}}$. To show this, use descent:

$$\pi^*\Omega^1_{X/S} \longrightarrow \Omega^1_{Y/S} \longrightarrow \Omega^1_{Y/X} = \Omega^1_{\pi'} = \pi^*\Omega^1_{X/\mathfrak{X}}$$

Thus we get $\Omega^1_{X/S} \to \Omega^1_{X/\mathfrak{X}}$.

We note that, for Artin stacks, this is usually not surjective. But it is surjective for DM stacks.

Theorem 4.4.7

$\mathfrak{X}$ is DM iff $\Delta_{\mathfrak{X}/S}$ is formally unramified.

Recall that in the above, for smooth $X \twoheadrightarrow \mathfrak{X}$ we defined $\Omega^1_{X/\mathfrak{X}}$ coherent locally free. We also mentioned the idea of the proof, which is to look at where $\Omega_{X/\mathfrak{X}}$ is free, then we get

$$\begin{tikzcd} X' \arrow[d, two heads, "et"] & \mathbb{A}^r_{\mathfrak{X}} \arrow[d, two heads, "g"', "et"] \\ X \arrow[r, two heads, "sm"] & \mathfrak{X} \end{tikzcd}$$

Then using $\Delta_{\mathfrak{X}/S}$ is formally unramified, we will "slice" g until it becomes relative dim 0, i.e. étale.

Now we start the proof.

Proof. We start with the easy direction. Assume $\mathfrak{X}$ is DM. Choose étale cover $X \twoheadrightarrow \mathfrak{X}$.

Consider the following diagram

$$\begin{tikzcd}
X \times_{\mathfrak{X}} X \arrow[r, "b"] \arrow[d, two heads, "et"] \arrow[dr, phantom, "\square"] & X \times_S X \arrow[d, two heads, "et"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times_S \mathfrak{X}
\end{tikzcd}$$

The goal is to show b is formally unramified. Let $X \times_S X \to X$ be the projection, then we get the following diagram

$$\begin{tikzcd}
X \times_{\mathfrak{X}} X \arrow[r, "\pi \circ b"] \arrow[d, "\pi \circ b"] & X \arrow[d, two heads] \\
X \arrow[r, two heads, "et"] & \mathfrak{X}
\end{tikzcd}$$

However, note $\pi \circ b$ is étale, hence b is unramified, and hence Δ is unramified as desired.

Conversely, suppose Δ is formally unramified. Let $k = \overline{k}$ be a ACF, $y \in \mathfrak{X}(k)$. Choose $p : X \xrightarrow{sm} \mathfrak{X}$ with X affine, such that

$$\begin{tikzcd}
\emptyset \neq X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} k \arrow[d, "y"] \\
X \arrow[r, "p"] & \mathfrak{X}
\end{tikzcd}$$

However, since k is ACF, we get a section x':

$$\begin{tikzcd}
X_y \arrow[r, shift right] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} k \arrow[l, bend right, "x'"'] \arrow[d, "y"] \\
X \arrow[r, "p"'] & \mathfrak{X}
\end{tikzcd}$$

Let k_0 be the residue field of image of x' in S and let k_0^{sep} be the separable closure of k_0.

Our goal is to show étale locally on S and X, we will find $W \subseteq X$ closed such that $W \xrightarrow{\text{ét}} \mathfrak{X}$ with $W_y \neq \emptyset$. Then we are done as $\coprod_y W_y \twoheadrightarrow \mathfrak{X}$ is étale.

Now we get the diagram

$$\begin{tikzcd}
Z \arrow[r, "\pi"] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "p"] \\
X \arrow[r, "p"] & \mathfrak{X}
\end{tikzcd}$$

which is the same as the following diagram

$$\begin{tikzcd}
Z \arrow[r, "{(\pi,\pi')}"] \arrow[d] \arrow[dr, phantom, "\square"] & X \times_S X \arrow[d] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

Now let $\Omega'_{\pi'} := \Omega^1_{Z/X} = \pi^*\Omega^1_{X/\mathfrak{X}}$. Last time we also constructed $\Omega^1_{X/S} \to \Omega^1_{X/\mathfrak{X}}$. This is usually not surjective, but we will show it is the case for DM stacks. To show it is surjective, by descent, it is enough to show surjective after applying π^*. We see Δ is formally unramified implies (π, π') is formally unramified, and hence

$$(\pi, \pi')^*\Omega_{X\times_S X/S} = \pi^*\Omega^1_{X/S} \oplus (\pi')^*\Omega_{X/S} \twoheadrightarrow \Omega_{Z/S}$$

This gives

$$\begin{tikzcd}
\pi^*\Omega^1_{X/S} \arrow[r, hook] \arrow[ddr, "\pi^*\phi"'] & \pi^*\Omega^1_{X/S} \oplus (\pi')^*\Omega^1_{X/S} \arrow[d] \arrow[dd, bend left, "0"] \\
 & \Omega^1_{Z/S} \arrow[d] \\
 & \Omega^1_{\pi'} = \pi^*\Omega^1_{X/\mathfrak{X}}
\end{tikzcd}$$

Since we have that 0 map, it gives us that $\pi^*\phi$ is surjective, and hence ϕ is surjective.

We have

$$\begin{tikzcd}
\mathscr{O}_X, \arrow[r, "d"] \arrow[dr] & \Omega^1_{X/S} \arrow[d, two heads] \\
 & \Omega^1_{X/\mathfrak{X}}
\end{tikzcd}$$

where we call the arrow $\mathscr{O}_X \to \Omega^1_{X/\mathfrak{X}}$ d as well. Locally, $d : \mathscr{O}_X \to \Omega^1_{X/S}$ has image generates, so the same is true for $d : \mathscr{O}_X \to \Omega^1_{X/\mathfrak{X}}$.

Since $\Omega^1_{X/\mathfrak{X}}$ is locally free, so we need to look étale locally where $\Omega^1_{X/\mathfrak{X}}$ is free and there exists $f_1, \ldots, f_r \in \Gamma(\mathscr{O}_U)$ such that $df_1, \ldots, df_r$ is a basis for $\Omega^1_{X/\mathfrak{X}}|_U$.

Shrinking X, we may assume $U = X$. Let $f_i \in \Gamma(\mathscr{O}_X)$ that df_i generates $\Omega^1_{X/\mathfrak{X}}$, so we get $F : X \xrightarrow{(p, f_1, \ldots, f_r)} \mathfrak{X} \times_S \mathbb{A}^r_S$. Then,

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, "sm"'] & \mathfrak{X} \times_S \mathbb{A}^r_S = \mathbb{A}^r_{\mathfrak{X}} \arrow[dl, "sm"] \\
\mathfrak{X} &
\end{tikzcd}$$

and $\dim_{x'}(X) = \dim_{x'}(\mathbb{A}^r_{\mathfrak{X}})$ because $\Omega'_{X/\mathfrak{X}}$ is free for rank r.

F is smooth map, representable and relative dim 0 in neighbourhood of x', so it is étale in neighbourhood of x'. Shrink to assume F is étale. Then $F_y : X_y \xrightarrow{\text{ét}} \mathbb{A}^r_k$. Then F_y is étale implies the image is open. Let $f \in k[t_1, \ldots, t_r]$ such that $\emptyset \neq D(f) \subseteq F_y(X_y)$. Over k_0^{sep}, there exists $a_1, \ldots, a_r \in k_0^{sep}$ such that $f(a_1, \ldots, a_r) \neq 0$.

In particular, we see

$$\begin{tikzcd}
(a_1, \ldots, a_r) \in \arrow[d, mapsto] & \mathbb{A}_{k_0^{sep}} \arrow[d] \\
\text{closed point } Q & \mathbb{A}^r_{k_0}
\end{tikzcd}$$

Now, we import a fact, which is that, we can take an étale neighbourhood $S' \to S$ of x' such that there exists closed E with $E \subseteq \mathbb{A}^r_{S'}$ and diagram

$$\begin{tikzcd}
E \arrow[r, "\subseteq"] \arrow[dr, "et"'] & \mathbb{A}^r_{S'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{A}^r_S \arrow[d] & \operatorname{Spec} k_0^{sep} \arrow[l, "Q"'] \arrow[d] \\
 & S' \arrow[r, "et"'] & S & \operatorname{Spec} k_0 \arrow[l, "x'"]
\end{tikzcd}$$

then there exists unique arrow $\operatorname{Spec} k \to S'$ with $Q \in E$, i.e. we get the following diagram

$$\begin{tikzcd}
E \arrow[r, "\subseteq"] \arrow[dr, "et"'] & \mathbb{A}^r_{S'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{A}^r_S \arrow[d] & \operatorname{Spec} k_0^{sep} \arrow[l, "Q"'] \arrow[d] \\
 & S' \arrow[r, "et"'] & S & \operatorname{Spec} k_0 \arrow[l, "x'"] \arrow[ll, dashed, bend left, "\exists"]
\end{tikzcd}$$

Therefore, we see

$$\begin{tikzcd}
W \arrow[r, "\subseteq", "closed"'] \arrow[d, "et"'] \arrow[dr, phantom, "\square"] & X \times_S \mathbb{A}^r_{S'} \arrow[d, "F\times \mathrm{Id}"'] \\
\mathfrak{X} \times_S E \arrow[r, "\subseteq"'] & \mathfrak{X} \times_{S'} \mathbb{A}^r_{S'}
\end{tikzcd}$$

Therefore, $W_y \neq \emptyset$ by construction and hence

$$\begin{tikzcd}
W \arrow[r, "et"] & E \times_{S'} \mathfrak{X} \arrow[r, "et"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\
 & E \arrow[r, "et"'] & S'
\end{tikzcd}$$

Now $W \to \mathfrak{X}$ is our desired map.

Corollary 4.4.7.1

If G is a finite group, then $\mathfrak{X} = [X/G]$ DM.

Proof. We just need Aut groups are finite over $k = \overline{k}$. Over $k = \overline{k}$, G-torsors are trivial, so our point is

$$\begin{tikzcd}
G \arrow[r, "G-equiv", "\alpha"'] \arrow[d] & X \\
\operatorname{Spec} k &
\end{tikzcd}$$

and Aut group is

$$\begin{tikzcd}
G \arrow[dr, "\gamma", "\sim"'] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right] & & \\
& G \arrow[r, "\alpha"] \arrow[d] & X \\
& \operatorname{Spec} k &
\end{tikzcd}$$

So $\gamma(1) = g$. Then $\gamma(h) = h \circ \gamma(1) = hg$ and hence we get

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[d, mapsto] & \alpha(1) \arrow[d, mapsto, "="] \\
\gamma(1) \arrow[r, mapsto] & \gamma(1) \cdot \alpha(1)
\end{tikzcd}$$

In other word, Aut is stablizer of $\alpha(1)$. Hence we get

$$\begin{tikzcd}
\mathrm{Aut} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} k \arrow[d] \\
G \arrow[r] & X
\end{tikzcd}$$

$$g \longmapsto g \cdot \alpha(1)$$

and so Aut is finite as desired.

At the start of the course, we talked about 5 general points determine conic. We did this through geometry on moduli space. In particular, moduli space of singular conics is exactly $\mathbb{P}^5$ (if we drop singular, then this is not true!).

Even if we only interested in smooth curves, i.e. M_g. To do intersection theory, we need a compactification $\overline{M_g}$. So, we need a notion of properness. To do this, we need quasi-coherent sheaves on $\mathfrak{X}$.

Theorem 4.4.8: Eisenbud-Harris

It is impossible to write down a general $g \geq 24$ curve.

This uses intersection theory on $\overline{M_g}$. Note here we are asking for general $g \geq 24$ curve. To see what this means, we consider the example of elliptic curve. For that, we know the short form for elliptic curve is given by $y^2 = x^3 + ax + b$. Thus it is the same as a dominant rational map $\mathbb{A}^2_{a,b} \dashrightarrow M_{1,1}$, i.e. $\mathbb{A}^2_{a,b} \backslash (\text{discriminant} = 0) \to M_{1,1}$. They showed M_g is of general type.

There is also another similar problem, where we look at $\mathscr{A}_g$, the moduli space of dim g abelian varities. Then we know $\mathscr{A}_{\geq 7}$ is of general type, $\mathscr{A}_{\leq 5}$ is , and $\mathscr{A}_6$ is unknown.

Finally, let us record a theorem about coverings of DM stacks.

Theorem 4.4.9: Le Lemme de Gabber

Let $\mathfrak{X}$ be a DM stack separated and of finite type over a Noetherian scheme S. Then there is a finite, generically étale, and surjective morphism $Z \to \mathfrak{X}$ from a scheme Z.

Proof. A proof can be found in Angelo Vistoli's "Intersection theory on algebraic stacks and on their moduli spaces" in 1989. It is Proposition 2.6.

4.5 Quasi-coherent Sheaves

Now we jump to quasi-coherent/coherent sheaves on stack.

Definition 4.5.1

We define ***Lisse-étale site*** Lis-et$(\mathfrak{X})$ on $\mathfrak{X}$ as follows (the topos is denoted by $\mathfrak{X}_{\text{Lis-et}}$). The site Lis-et$(\mathfrak{X})$ has objects $T \xrightarrow[sm]{t} \mathfrak{X}$ with T being scheme, and morphisms

$$\begin{tikzcd} T' \arrow[r, "f"] \arrow[dr, "t'"'] & T \arrow[d, "t"] \arrow[dl, phantom, "\overset{f^\flat}{\Rightarrow}"] \\ & \mathfrak{X} \end{tikzcd}$$

The coverings are given by family of diagrams of the form

$$\begin{tikzcd} T_i \arrow[r, "f_i"] \arrow[dr, "t_i"'] & T \arrow[d, "t"] \arrow[dl, phantom, "\Rightarrow"] \\ & \mathfrak{X} \end{tikzcd}$$

such that we have étale surjection $\coprod T_i \twoheadrightarrow T$.

Then, we define $\mathscr{O}_{\mathfrak{X}} \in \mathfrak{X}_{\text{Lis-et}}$ as $\mathscr{O}_{\mathfrak{X}}(T \xrightarrow{t} \mathfrak{X}) := \Gamma(\mathscr{O}_T)$.

Now let $\mathcal{C}$ be the following category. The objects are: for all $(T, t) \in$ Lis-et$(\mathfrak{X})$ a choice of étale sheaf of sets $\mathcal{F}_{(T,t)} \in T_{\text{ét}}$ and for all $(f, f^\flat) : (T', t') \to (T, t)$ a choice

$$f^{-1}\mathcal{F}_{(T,t)} \xrightarrow{\rho_{(f,f^\flat)}} \mathcal{F}_{T',t'}$$

such that:

1. if $f : T' \to T$ is étale then $\rho_{(f,f^\flat)}$ is isomorphism.

2. For diagram

$$\begin{tikzcd} (T'', t'') \arrow[r, "(g,g^\flat)"] \arrow[dr] & (T', t') \arrow[d, "(f,f^\flat)"] \\ & (T, t) \end{tikzcd}$$

we have

$$\begin{tikzcd} g^{-1}f^{-1}\mathcal{F}_{(T,t)} \arrow[r, "g^{-1}\rho_{(f,f^\flat)}"] \arrow[d, "\text{can}"'] & g^{-1}\mathcal{F}_{(T',t')} \arrow[d, "\rho_{(g,g^\flat)}"] \\ (fg)^{-1}\mathcal{F}_{(T,t)} \arrow[r, "\rho_{(f,f^\flat)\circ(g,g^\flat)}"'] & \mathcal{F}_{(T'',t'')} \end{tikzcd}$$

A morphism between $(\{\mathcal{F}_{(T,t)}\}, \{\rho_{(f,f^\flat)}\}) \to (\{\mathcal{G}_{(T,t)}\}, \{\lambda_{(f,f^\flat)}\})$ in $\mathcal{C}$ is a collection of morphisms $\gamma_{(T,t)} : \mathcal{F}_{(T,t)} \to \mathcal{G}_{(T,t)}$ so the following diagram commutes

$$\begin{tikzcd} f^{-1}\mathcal{F}_{(T,t)} \arrow[r, "f^{-1}\gamma_{(T,t)}"] \arrow[d, "\rho_{(f,f^\flat)}"] & f^{-1}\mathcal{G}_{(T,t)} \arrow[d, "\lambda_{(f,f^\flat)}"] \\ \mathcal{F}_{(T',t')} \arrow[r, "\gamma_{(T',t')}"] & \mathcal{G}_{(T',t')} \end{tikzcd}$$

We have $\mathcal{C} \to \mathfrak{X}_{\text{Lis-et}}$ given by $(\{\mathcal{F}_{(T,t)}\}, \{\rho_{(f,f^\flat)}\})$ maps to the sheaf $\mathcal{F}$ given by $\mathcal{F}(T \xrightarrow{t} \mathfrak{X}) = \mathcal{F}_{(T,t)}(T)$.

This is a presheaf where transition maps of $\mathcal{F}$ come from $\rho_{(f,f^\flat)}$.

It is a sheaf because étale covers in Lis-et$(\mathfrak{X})$ are already coverings in $T_{\text{ét}}$.

Conversely, $\mathfrak{X}_{\text{Lis-et}} \to \mathcal{C}$ given by $\mathcal{F}$ maps to the object $\mathcal{F}_{(T,t)} := \mathcal{F}|_{T_{\text{ét}}}$.

Therefore, we get $\mathcal{C} \cong \mathfrak{X}_{\text{Lis-et}}$.

Definition 4.5.2

Let Λ be a sheaf of rings on Lis-et$(\mathfrak{X})$. Let $\mathcal{F}$ be a sheaf of Λ-module. Then:

1. We say $\mathcal{F}$ is ***Cartesian*** if for all $(T', t') \to (T, t)$, we get

$$f^*\mathcal{F}_{(T,t)} := f^{-1}\mathcal{F}_{(T,t)} \otimes_{f^{-1}\Lambda_{(T,t)}} \Lambda_{(T',t')} \xrightarrow{\sim} \mathcal{F}_{(T',t')}$$

2. We say $\mathcal{F}$ is ***quasi-coherent*** if it is Cartesian $\mathcal{O}_{\mathfrak{X}}$-module and for all (T, t), we have $\mathcal{F}_{(T,t)} \in (\mathbf{Qcoh})(T_{\text{ét}})$ is quasi-coherent.
3. We say $\mathcal{F}$ is ***coherent*** if $\mathcal{X}$ is locally Noetherian (note this implies for anyu $(T, t) \in \text{Lis-et}(\mathfrak{X})$ we get T locally Noetherian), $\mathcal{F}$ is quasi-coherent and for all (T, t), we have $\mathcal{F}_{(T,t)} \in (\mathbf{Coh})(T_{\text{ét}})$.

We defined the Lis-et site. We also gave an alternative description, namely if $\mathcal{F} \in \mathfrak{X}_{\text{Lis-et}}$ is a sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules, then for all $t : T \xrightarrow{sm} \mathfrak{X}$ we let $\mathcal{F}_{(T,t)} := \mathcal{F}|_{T_{\text{ét}}}$.

Given any

$$\begin{tikzcd} T' \arrow[r, "f"] \arrow[dr, "t'"'] & T \arrow[d, "t"] \arrow[dl, phantom, "\Rightarrow"] \\ & \mathfrak{X} \end{tikzcd}$$

we have a module-theoretic pullback $f^*\mathcal{F}_{(T,t)} := f^{-1}\mathcal{F}_{(T,t)} \otimes_{f^{-1}\mathcal{O}_T} \mathcal{O}_{T'}$. In particular we get natural map $f^*\mathcal{F}_{(T,t)} \to \mathcal{F}_{(T',t')}$.

We also defined Cartesian $\mathcal{F}$, which is that, if for all $T' \to T$ we get $f^*\mathcal{F}_{(T,t)} \xrightarrow{\sim} \mathcal{F}_{(T',t')}$. We say $\mathcal{F}$ is quasi-coherent if $\mathcal{F}$ is Cartesian and all $\mathcal{F}_{(T,t)}$ are quasi-coherent. If $\mathfrak{X}$ is locally Noetherian, then $\mathcal{F}$ is coherent if all $\mathcal{F}_{(T,t)}$ are coherent.

This is a lot of data to keep track of, but as you would expect, to check this, we only need to do it on an open cover.

Proposition 4.5.3

Let $X \twoheadrightarrow \mathfrak{X}$ be smooth covering and $\mathcal{F}$ Cartesian. Then $\mathcal{F}$ is quasi-coherent iff $\mathcal{F}_{(X,x)}$ is quasi-coherent. If $\mathfrak{X}$ is locally Noetherian, then $\mathcal{F}$ is coherent iff $\mathcal{F}_{(X,x)}$ is coherent.

Proof. For all $Y \xrightarrow{y} \mathfrak{X}$, take pullback

$$\begin{tikzcd} Z \arrow[r, "p"] \arrow[d, "sm"', "\pi"] \arrow[dr, phantom, "\square"] & X \arrow[d, "sm"', "x"] \\ Y \arrow[r, "y"] & \mathfrak{X} \end{tikzcd}$$

We see $\pi^*\mathcal{F}_{(Y,y)} \cong \mathcal{F}_{(Z,z\pi)}$ because $\mathcal{F}$ is Cartesian. But $\mathcal{F}_{(Z,y\pi)} \cong p^*\mathcal{F}_{(X,x)}$ where $\mathcal{F}_{(X,x)}$ is quasi-coherent, hence $\mathcal{F}_{(Z,z)}$ is quasi-coherent. Therefore $\mathcal{F}_{(Y,y)}$ is by descent quasi-coherent.

The argument for coherent is similar.

If $\mathfrak{X}$ is DM, then we have an étale site $\text{Ét}(\mathfrak{X})$ defined as follows (it always exists, but in general we don't have a good theory).

The objects are $T \xrightarrow[et]{t} \mathfrak{X}$, and morphisms are

$$\begin{tikzcd} T' \arrow[r, "et"] \arrow[dr, "et"'] & T \arrow[d, "et"] \arrow[dl, phantom, "\Leftarrow"] \\ & \mathfrak{X} \end{tikzcd}$$

The coverings are just étale coverings.

You can define Cartesian, quasi-coherent, and coherent in the same way as $\mathfrak{X}_{\text{Lis-et}}$ for $\mathfrak{X}_{\text{ét}}$.

From now on, if we are talking about $\mathfrak{X}_{\text{ét}}$, we always assume $\mathfrak{X}$ is DM.

In particular, the notion of (quasi)-coherent on $\mathfrak{X}_{\text{ét}}$ agrees with $\mathfrak{X}_{\text{Lis-et}}$. In particular, we get the restriction map $r : \text{Ét}(\mathfrak{X}) \to \text{Lis-et}(\mathfrak{X})$ given by $(T \xrightarrow[et]{f}) \mapsto (T \xrightarrow[sm]{f} \mathfrak{X})$. Thus we get $r_* : (\mathbf{Qcoh})(\mathfrak{X}_{\text{Lis-et}}) \to (\mathbf{Qcoh})(\mathfrak{X}_{\text{ét}})$.

Proposition 4.5.4

r_* is an equivalence. It is also equivalence for coherent sheaves if $\mathfrak{X}$ is locally Noetherian.

Proof. Start with $\mathcal{F} \in (\mathbf{Qcoh})(\mathfrak{X}_{\text{ét}})$, we will extend to $\tilde{\mathcal{M}} \in (\mathbf{Qcoh})(\mathfrak{X}_{\text{Lis-et}})$ in a unique way.

To do this, we choose a smooth cover $X \xrightarrow{\text{ét}} \mathfrak{X}$. Suppose we are given $Y \xrightarrow{sm} \mathfrak{X}$, we want to define $\tilde{\mathcal{M}}|_{Y_{\text{ét}}}$. Well, we pullback the arrow $Y \xrightarrow{sm} \mathfrak{X}$, namely

$$\begin{tikzcd} Z \arrow[r, "sm", "f"'] \arrow[d, two heads, "\text{ét}"] \arrow[dr, phantom, "\square"] & X \arrow[d, two heads, "et"] \\ Y \arrow[r, "sm"] & \mathfrak{X} \end{tikzcd}$$

By descent, it is enough to define $\tilde{\mathcal{M}} \in Z_{\text{ét}}$ with descent data. Because we want $\tilde{\mathcal{M}}$ to be Cartesian, so we need $\tilde{\mathcal{M}}_Z = f^*\tilde{\mathcal{M}}_X = f^*\mathcal{M}_X$. We get a diagram

$$\begin{tikzcd} Z' \arrow[r, "g"] \arrow[d, shift right, "\pi_2"'] \arrow[d, shift left, "\pi_1"] & X' \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\ Z \arrow[r, "f"'] \arrow[d, "\pi"'] & X \arrow[d, "p"'] \\ Y \arrow[r] & \mathfrak{X} \end{tikzcd}$$

Since $\mathcal{M}$ is Cartesian, so $p_1^*\mathcal{M}_X \cong p_2^*\mathcal{M}_X$. We want $\tilde{\mathcal{M}}$ to be Cartesian, so

$$\pi_1^*\tilde{\mathcal{M}}_Z \cong g^*p_1^*\mathcal{M}_X \cong g^*p_2^*\mathcal{M}_X \cong \pi_2^*\tilde{\mathcal{M}}_Z$$

Thus $\tilde{\mathcal{M}}_Z$ has descent data and it descends to $\tilde{\mathcal{M}}_Y$, as desired.

The next topic is differentials on stacks.

We have shown how to define $\Omega^1_{X/\mathfrak{X}}$ if we have smooth cover $X \twoheadrightarrow \mathfrak{X}$.

If $\mathfrak{X}$ is DM, you can even define $\Omega^1_{\mathfrak{X}} \in (\mathbf{Coh})(\mathfrak{X}_{\text{ét}})$. Well, to do this, consider

$$X' = X \times_{\mathfrak{X}} X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \xrightarrow[\pi]{} \!\!\!\!\rightarrow \mathfrak{X}$$

We then descend Ω^1_X to get $\Omega^1_{\mathfrak{X}}$. Here π_1, π_2 are étale, so $\pi_1^*\Omega^1_X \cong \Omega^1_{X'} \cong \pi_2^*\Omega^1_X$ are canonical isomorphisms, thus we indeed have descent data. Therefore, we get $\Omega^1_{\mathfrak{X}}$ such that $\pi^*\Omega^1_{\mathfrak{X}} = \Omega^1_X$.

For Artin stack, this does not work because all you get $\pi_1 : X' \xrightarrow{sm} X$ and you have a map

$$0 \longrightarrow \pi_1^*\Omega^1_X \longrightarrow \Omega^1_{X'} \longrightarrow \Omega^1_{X'/X} \longrightarrow 0$$

where $\Omega^1_{X'/X} \neq 0$ unless it is étale. Therefore we cannot descend Ω^1_X because we have no isomorphism $\pi_1^*\Omega^1_X \to \Omega^1_{X'}$.

There are some remediations.

For exmaple, there is something called pseudo-differentials, but in that theory, we get $\Omega^1_{\mathfrak{X}/\mathfrak{X}} \neq 0$, which is weird.

Another solution is the cotangent complex $L_{\mathfrak{X}}$. This is a 2-term complex such that $\pi^*L_{\mathfrak{X}} \cong [\Omega^1_X \to \Omega^1_{X/\mathfrak{X}}]$, where $\pi : X \twoheadrightarrow \mathfrak{X}$. If $\mathfrak{X} = [X/G]$, then we can say more. In particular, in this case we get $\Omega^1_{X/\mathfrak{X}} \cong \mathscr{O}_X \otimes \mathfrak{g}^*$ where $\mathfrak{g}^*$ is the dual of the Lie algebra of G, i.e. $g = \Omega^1_{G,e}$ where $e \in G$ is the identity point.

Example 4.5.5

We have seen (in the proofs above) that to define

$$\mathcal{F} \in (\mathbf{Qcoh})(\mathfrak{X}_{\text{ét}})$$

it is good enough to define $\mathcal{M} \in (\mathbf{Qcoh})(X)$ plus descent data where $\pi : X \twoheadrightarrow \mathfrak{X}$ is étale covering.

Let's look at $\mathfrak{X} = [X/G]$. In this case, we know $X \times_{\mathfrak{X}} X \cong G \times X$, and hence

$$X \times_{\mathfrak{X}} X \cong G \times X \underset{p}{\overset{\sigma}{\rightrightarrows}} X \xrightarrow{\pi} \mathfrak{X}$$

where σ is the action, and p is the projection, i.e. $(g, x) \mapsto gx$ and $(g, x) \mapsto x$. We want $\mathcal{M}$ quasi-coherent on X and $\sigma^*\mathcal{M} \cong p^*\mathcal{M}$ plus the cocycle condition. But that last part (i.e. $\sigma^*\mathcal{M} \cong p^*\mathcal{M}$ and cocycle condition) is just saying we have G acts on $\mathcal{M}$. Therefore, $(\mathbf{Qcoh})(\mathfrak{X}_{\text{ét}})$ is just quasi-coherent sheaves on $\mathfrak{X}_{\text{ét}}$ with G-linearization, i.e. G-action.

Example 4.5.6

For example, if $\mathfrak{X} = BG$, and we take $\operatorname{Spec} k \twoheadrightarrow \mathfrak{X}$, then $(\mathbf{Qcoh})(BG)$ is isomorphic to vector space plus G-action, which is G representations.

Example 4.5.7

If $G \circlearrowleft X$ acts freely, i.e. X/G is algebraic space (e.g. if $X = \operatorname{Spec} A$ then $X/G = \operatorname{Spec}(A^G)$). Then $(\mathbf{Qcoh})(X/G) = (\mathbf{Qcoh})(X)$ plus G-linearization

Example 4.5.8

Suppose $X = \operatorname{Spec} L$ where L/K is Galois with Galois group G. Then $[X/G] = X/G = \operatorname{Spec} K$. Therefore, we see K-vector spaces is the same thing as L-vector spaces with G-action.

In general, given $\mathcal{F} \in (\mathbf{Qcoh})([X/G])$, it corresponds to $\mathcal{M} \in (\mathbf{Qcoh})(X)$ with G-action. If X is affine, then the cohomology $H^i(\mathcal{F})$ is equal the group cohomology $H^i(G, \Gamma(\mathcal{M}))$. If X is not affine, there is a spectral sequence relating $H^i(\mathcal{F})$ with $H^p(G, H^q(\mathcal{M}))$.

In general, if $f : \mathfrak{X} \to \mathfrak{Y}$ is quasi-compact and quasi-separated, then $f_*\mathcal{F}$ quasi-coherent if $\mathcal{F}$ is quasi-coherent. In particular, we get $f_*\mathcal{F}(Y \xrightarrow{sm} \mathfrak{Y}) = \mathcal{F}(Y \times_{\mathfrak{Y}} \mathfrak{X} \xrightarrow{sm} \mathfrak{X})$. There is a left adjoint f^*, but we are not going to define it.

Definition 4.5.9

We say $f : \mathfrak{X} \to \mathfrak{Y}$ is ***affine*** if it is representable and affine.

Given $\mathscr{A}$ a sheaf of $\mathscr{O}_{\mathfrak{X}}$-algebra, we can define $\operatorname{Spec}(\mathscr{A}) \to \mathfrak{X}$ as follows: the T-points are given as follows:

$$\begin{tikzcd} & \operatorname{Spec}\mathscr{A} \arrow[d] \\ T \arrow[ur] \arrow[r] & \mathfrak{X} \end{tikzcd}$$

is a choice $\rho : t^*\mathscr{A} \to \mathscr{O}_T$ morphism of $\mathscr{O}_T$-algebras. By construction, we see we get

$$\begin{tikzcd} \operatorname{Spec}(t^*\mathscr{A}) \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \operatorname{Spec}\mathscr{A} \arrow[d, "p"] \\ T \arrow[r, "t"] & \mathfrak{X} \end{tikzcd}$$

So π is an affine map of schemes, and so p is representable and affine.

Moreover, $\operatorname{Spec}\mathscr{A}$ is a stack because $\mathscr{O}_X$-algebra satisfy descent. Hence $\operatorname{Spec}\mathscr{A}$ is an Artin stack.

Like for schemes, we have the bijective correspondence

$$\left\{ \begin{matrix} \text{affine} \\ \text{maps to } \mathfrak{X} \end{matrix} \right\} \longleftrightarrow \left\{ \mathscr{O}_X\text{-algebras} \right\}$$

given by $\mathscr{A} \mapsto (\operatorname{Spec}\mathscr{A} \to \mathfrak{X})$ and $(\mathfrak{Y} \xrightarrow{f} \mathfrak{X}) \mapsto f_*\mathscr{O}_{\mathfrak{Y}}$.

The next topic is closed substacks.

Definition 4.5.10

We say $\mathfrak{Z} \to \mathfrak{X}$ is ***closed/open immersion*** if it is representable and it is closed/open immersion.

Definition 4.5.11

Given $\mathfrak{X} \xrightarrow{f} Y$ with Y a scheme, the ***image*** of f is given by: choose $X \xrightarrow{\pi} \mathfrak{X} \xrightarrow{f} Y$, then $\operatorname{Im}(f) := \operatorname{Im}(f \circ \pi)$.

We say f is ***closed*** if for all $\mathfrak{Z} \subseteq \mathfrak{X}$ closed substacks, $\operatorname{Im}(\mathfrak{Z} \subseteq \mathfrak{X} \to Y)$ is closed.

Definition 4.5.12

We say $g : \mathfrak{X} \to \mathfrak{Y}$ is ***universally closed*** if for all $Y \to \mathfrak{Y}$ with Y scheme and diagram

$$\begin{tikzcd} \mathfrak{X} \times_{\mathfrak{Y}} Y \arrow[r] \arrow[d, "g_Y"] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "g"] \\ Y \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

we have g_Y is closed.

Definition 4.5.13

We say $f : \mathfrak{X} \to \mathfrak{Y}$ is ***proper*** if f is separated, of finite type, and universally closed.

Example 4.5.14

Let A be Abelian variety, then $BA \to \operatorname{Spec} k$ is proper (it is separated because A is proper).

Example 4.5.15

If G is finite group, then $BG \to \operatorname{Spec} k$ is proper.

4.6 Valuative Criteria

Theorem 4.6.1

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be finite type map of locally Noetherian Artin stacks.

1. *Then f is separated iff for all DVR R with $K = \operatorname{Frac}(R)$ and diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, dashed, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, shift left, "y"] \arrow[ur, shift right, "x"'] & \mathfrak{Y} \end{tikzcd}$$

   we have $x \cong y$.
2. *Then f is proper iff f is separated and for all DVR R with $K = \operatorname{Frac}(R)$, there*

exists finite extension K'/K and normalization of R in K' such that we have the dashed arrow in the following diagram

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[r, "\exists"] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, dashed, "f"] \\
\operatorname{Spec} R' \arrow[r, "\exists"'] \arrow[urr, dashed, "\exists"] & \operatorname{Spec} R \arrow[r] & \mathfrak{Y}
\end{tikzcd}$$

So, how do you think about this? To get some intuition, lets take $\mathfrak{Y}$ to be $\operatorname{Spec} k$. Then $f : \mathfrak{X} \to \operatorname{Spec} k$ is a moduli space, and the theorem says, f is proper if and only if, if we are given $x \in \mathfrak{X}(K)$, e.g. a curve over K, then we can extend it to $\mathfrak{X}(R)$, e.g. a curve over R, if we allow for finite extensions of K.

Example 4.6.2

Let's consider an exmaple where the "weak form" of valuative criteria does not hold, i.e. if we don't allow finite extension.

We know BG is proper if G is finite. Let's take $B(\mathbb{Z}/2)$ with $\operatorname{char} k = 0$. Then we have

$$\operatorname{Spec} K \longrightarrow B(\mathbb{Z}/2)$$

with $R = k[[t]]$ and $K = \operatorname{Frac}(R) = k((t))$. Then we have a $\mathbb{Z}/2$-torsor $\operatorname{Spec} k((\sqrt{t})) \to \operatorname{Spec} K$ with the action being $\sqrt{t} \mapsto -\sqrt{t}$.

Then, we can't extend this to a $\mathbb{Z}/2$-torsor over R. We will not give a proof, but let's check the obvious choice does not work. The obvious choice would be extend to $\operatorname{Spec} k[[\sqrt{t}]] \to \operatorname{Spec} k[[t]]$, where we still have $\mathbb{Z}/2$-action extends to $\sqrt{t} \mapsto -\sqrt{t}$.

However, we have a problem: this is not a torsor because it is ramified over $t = 0$. We see $k[[\sqrt{t}]] = k[[t]][x]/(x^2 - t)$ and hence if $t = 0$ we get $k[[t]][x]/x^2$ which is not reduced and so $\operatorname{Spec} k[[\sqrt{t}]] \to \operatorname{Spec} k[[t]]$ is not etale. Thus it is not a torsor.

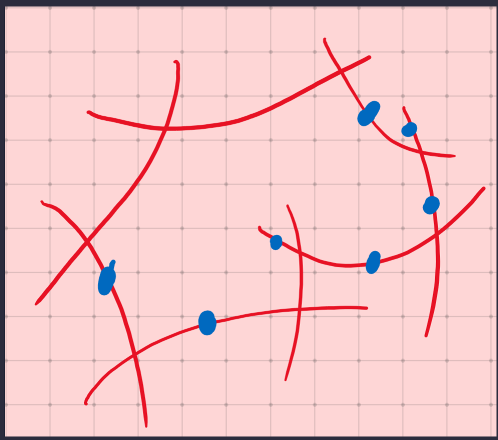
However, if we take finite extension $K' = k((\sqrt{t}))$ over K (this is degree 2 extension), then the torsor over K' is trivial, i.e. $\mathbb{Z}/2 \times \operatorname{Spec} K' \to \operatorname{Spec} K'$, in other word, we get

$$\begin{tikzcd}
\mathbb{Z}/2 \times \operatorname{Spec} R' \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{Z}/2 \times \operatorname{Spec} K' \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K' \arrow[d] \\
\operatorname{Spec} R' & \operatorname{Spec} K' \arrow[l] \arrow[r] & \operatorname{Spec} K
\end{tikzcd}$$

and now it extends.

Example 4.6.3

Deligne Mumford introduced a compactification $\overline{M}_{g,n}$ of $M_{g,n}$, i.e. $M_{g,n} \subseteq \overline{M}_{g,n}$ dense open substack and $\overline{M}_{g,n}$ is proper. We have $\overline{M}_{g,n}$ is like $M_{g,n}$ but the geometric fibers are “stable curves”, i.e. nodel curves which look like:



where each line is a curve, and intersections are nodes, and dots are marked points with the property that every genus 1 component has ≥ 1 special points and every genus 0 component has ≥ 3 special points, where we say a point is special if it is a node or a marked point.

This ensures automorphism group is finite. As a result, $\overline{M}_{g,n}$ is DM.

Next, how do we show $\overline{M}_{g,n}$ is proper? We use valuative criteria, but a stronger version of it. That is,

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & M_{g,n} \arrow[d, "\subseteq"] \\ \operatorname{Spec} R \arrow[r, dashed, "\exists!"'] & \overline{M}_{g,n} \end{tikzcd}$$

where we can get the dashed arrow after finite extension. Viz, we start with genus g curves and n marked points over K, we want to extend to stable curve over R, after finite extension.

Let’s consider the basic case: consider $\overline{M}_{1,1}$, i.e. we have elliptic curves over $\mathbb{Q}$. A theorem of Tate says it is impossible to extend to an elliptic curve over $\mathbb{Z}$, we always have a node (i.e. semi-stable reduction) or worth (e.g. cusp, which is called additive reduction).

Tate’s algorithm tells you how to make a finite extension so that you end up with semi-stable reduction.

In general, extending from K-points of $M_{g,n}$ to R-points of $\overline{M}_{g,n}$ is called semi-stable reduction theorem.

David Smyth in 2010 constructed “alternative compactification” of $M_{g,n}$. It is a different stack where you replace the $g = 1$ constrain with cusps.

So, why we even care about this?

This arises naturally when you run minimal model program (MMP) on coarse space of $\overline{M}_{g,n}$.

This should be enough motivation, and let's give a sketch proof of the theorem.

Proof. For separatedness: Suppose we have

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, dashed, "f"] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, shift left, "x"] \arrow[ur, shift right, "x'"'] & \mathfrak{Y}
\end{tikzcd}$$

then note we can base change to $\mathfrak{X}_R := \mathfrak{X} \times_{\mathfrak{Y}} R$ and get

$$\begin{tikzcd}
\mathfrak{X}_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X}_R \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} R \arrow[u, bend left, "x"] \arrow[u, bend right, "x'"']
\end{tikzcd}$$

But then this means we get

$$\begin{tikzcd}
\operatorname{Isom}(x_K, x'_K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K \arrow[d, dashed, "x_K"] & \operatorname{Spec} R \arrow[d, dashed, "x"] \arrow[dr, phantom, "\square"] & \operatorname{Isom}(x, x') \arrow[l] \arrow[d] \\
\operatorname{Spec} K \arrow[r, "x'_K"] & \mathfrak{X}_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X}_R \arrow[d] & \operatorname{Spec} R \arrow[l, "x'"'] \\
& \operatorname{Spec} K \arrow[r] & \operatorname{Spec} R \arrow[u, bend left, "x"] \arrow[u, bend right, "x'"'] &
\end{tikzcd}$$

where $x_K \cong x'_K$ means we have K-points of $\operatorname{Isom}(x, x')$. This extending to R-point just means $\operatorname{Isom}(x, x')$ is proper for all x, x'. Equivalent to $\Delta_{\mathfrak{X}}$ is proper but by definition this means $\mathfrak{X}$ is separated.

Now we prove the valuative criteria for properness: If f is proper, then it satisfies the valuative criteria. Again, pulling back to $\operatorname{Spec} R$, we have

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[d, "x"] \arrow[dd, bend right=60, "\mathrm{Id}"'] & \\
\mathfrak{X}_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X}_R \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} R
\end{tikzcd}$$

where $\operatorname{Spec} K \xrightarrow{x} \mathfrak{X}_K$ is a point and $\mathfrak{X}_K \to \operatorname{Spec} K$ is proper and hence x is a closed immersion. Therefore, we can take $\mathfrak{Z}$ to be the closure of x in $\mathfrak{X}_R$ and we get

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d, "x"] \arrow[dd, bend right=60, "\mathrm{Id}"'] & \mathfrak{Z} \arrow[d, "\subseteq"] \\
\mathfrak{X}_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X}_R \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} R
\end{tikzcd}$$

but since we have no extra irreducible components of $\mathfrak{Z}$ over closed point of $\operatorname{Spec} R$ because $\mathfrak{Z}$ is the closure of x, we see the arrow $\mathfrak{Z} \to \operatorname{Spec} R$ is flat, i.e.

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d, "x"] \arrow[dd, bend right=40, "\mathrm{Id}"'] & \mathfrak{Z} \arrow[d, "\subseteq"] \arrow[dd, bend left=40, "flat"] \\
\mathfrak{X}_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X}_R \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} R
\end{tikzcd}$$

Now choose a smooth cover $Z \twoheadrightarrow \mathfrak{Z}$ and we note

$$\begin{tikzcd}
Z \arrow[r, two heads, "sm"] \arrow[dr, "flat{,}surj"'] & \mathfrak{Z} \arrow[d] \\
& \operatorname{Spec} R
\end{tikzcd}$$

and hence

$$\begin{tikzcd}
& Z \arrow[d] \\
\operatorname{Spec} R' \arrow[r] \arrow[ur, dashed, "\exists"] & \operatorname{Spec} R
\end{tikzcd}$$

with K'/K finite extension.

Conversely, assume valuative criteria holds. Then consider

$$\begin{tikzcd}
\mathfrak{X}_Y \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "f"] \\
Y \arrow[r, two heads, "sm"] & \mathfrak{Y}
\end{tikzcd}$$

by descend, we just need to show g is proper, so we may as well assume $\mathfrak{Y} = Y$.

Now note by Chow's lemma, which says there eixsts proper morphism $p : P \twoheadrightarrow \mathfrak{X}$, because $\mathfrak{X}$ is separated. Thus we get

$$\begin{tikzcd}
P \arrow[r, two heads, "proper", "p"'] \arrow[dr, "h"] & \mathfrak{X} \arrow[d, "f"] \\
& Y
\end{tikzcd}$$

and it is enough to show h is proper.

However, note p, h are representable, so proper iff we have valuative criteria, i.e. we have valuative criteria for f, p and hence we have valuative criteria for h (to see this, note we get

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[dd] & P \arrow[d, "p"] \\
& \mathfrak{X} \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] \arrow[uur, dashed] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where the $\operatorname{Spec} R \to \mathfrak{X}$ arrow is due to val crit for f and the $\operatorname{Spec} R \to P$ arrow is due to val crit for p) and hence h is proper.

4.7 Coarse Moduli Space & Local Structure For DM Stacks

The next topic is coarse moduli space, and in the process of proving this, we will need to study the local structure of DM stacks.

Definition 4.7.1

Let $\mathfrak{X}$ be Artin stack, X be algebraic space. Then $\pi : \mathfrak{X} \to X$ is ***coarse (moduli) space*** if:

1. for all $\mathfrak{X} \to Y$ with Y algebraic space, we get

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\pi"] \arrow[dr, "\forall"'] & X \arrow[d, dashed, "\exists!"] \\
& Y
\end{tikzcd}$$

2. π induces a bijection between $\mathfrak{X}(k)/\mathrm{iso} \to X(k)$ for all $k = \overline{k}$.

Example 4.7.2

Let G be a group scheme over k. Then $BG \xrightarrow{k} \operatorname{Spec} k$ with π the structure map is a coarse space map. To see this, note we have

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[d, two heads] \arrow[dr, "\mathrm{Id}"] \arrow[dd, bend right, "y"'] & \\
BG \arrow[d] & \operatorname{Spec} k \arrow[dl, dashed, "y"', "\exists!"] \\
Y &
\end{tikzcd}$$

where we recall $BG = [\operatorname{Spec} k/G]$. On the other hand, if $\Omega = \overline{\Omega}$ is ACF, then $BG(\Omega)/\mathrm{iso}$ only has one point, namely the trivial torsor, thus we only have one map $\operatorname{Spec} \Omega \to \operatorname{Spec} k$.

Example 4.7.3

If G is a finite group with $G \circlearrowleft \operatorname{Spec} A$, then the invariant map $\mathfrak{X} = [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is coarse space map.

Example 4.7.4

The coarse space of $M_{1,1}$ is $\mathbb{A}^1$, where the coarse space map is sending E to its j-invariant, i.e. $E \mapsto j(E)$.

Example 4.7.5

Let's consider a non-example. Say we have $\mathbb{G}_m \circlearrowleft \mathbb{A}^1$ with $\lambda \cdot x = \lambda x$. Then $[\mathbb{A}^1/\mathbb{G}_m]$ has 2 points. It has a $\mathbb{G}_m$-stablizer 0 and a trivial stablizer η. In particular, $0 \in \overline{\eta}$. We note $\mathfrak{X}$ has no coarse space as if $\pi : \mathfrak{X} \to X$ is a map into algebraic space, then $\pi(\eta)$ and $\pi(0)$ are both k-points, i.e. they have the same residue field. But then we also have $\pi(0) \in \overline{\pi(\eta)}$, it just doesn't happen.

In what follows, we will work towards prove the Keel-Mori theorem, which states roughly that corase moduli space exists for DM stacks. To do this, we will first need to study the local structure of DM stacks, which tells us we have étale neighbourhood $[\operatorname{Spec} A_i/G] \to \mathfrak{X}$, then show $\operatorname{Spec} A_i^G$ glues in étale topology to a coarse moduli space.

To begin with, we note quotient stacks forms a very rich source of DM stacks. In particular, we will see every DM stack is étale locally isomorphic to a quotient of the form $[\operatorname{Spec} A/G]$.

Definition 4.7.6

If G is a finite group acting on an algebraic space U, a ***G*-invariant morphism** $U \to X$ is a ***geometric quotient*** if:

1. for every algebraically closed field k, the map $U \to X$ induces a bijection $U(k)/G \xrightarrow{\sim} X(k)$, and
2. $U \to X$ is universal for G-invariant maps to algebraic spaces, i.e. every G-invariant map $U \to Y$ to an algebraic space factors uniquely through $U \to X$.

If G is fintie group acting on $\operatorname{Spec} A$, then $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient, as we will see later (this is a standard fact for schemes, but our definition is for algebraic spaces, so it is more complicated).

Example 4.7.7

Assume $\operatorname{char}(k) \neq 2$, then let $G = \mathbb{Z}/2$ acts on $\mathbb{A}^1$ via $-1 \cdot x = -x$. Then $k[x]^G = k[x^2]$, and the geometric quotient is the map $\mathbb{A}^1 = \operatorname{Spec} k[x] \to \operatorname{Spec} k[x^2] = \mathbb{A}^1$ given by sending x to x^2. On the other hand, as we already seen, let $G = \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{A}^2$ by $-1(x,y) = (-x,-y)$, then we get $k[x,y]^G = k[x^2, xy, y^2]$.

Theorem 4.7.8

If G is finite group acting on affine scheme $\operatorname{Spec} A$, then $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient. If A is f.g. over Noetherian ring R, then A^G is also f.g. over R.

Proof. We will only give an incomplete proof, in particular we only give sketch to the surjectivity part at the end.

Consider commutative diagram

$$\begin{tikzcd} U = \operatorname{Spec} A \arrow[d] \arrow[dr, "\tilde{\pi}"] & \\ \mathfrak{X} = [U/G] \arrow[r, "\pi"] & X = \operatorname{Spec} A^G \end{tikzcd}$$

Since $\tilde{\pi}$ is integral and dominant, it is surjective. To see $\tilde{\pi}$ is injective on G-orbits of geometric points, let k be ACF and $x, x' \in U(k)$ with $\tilde{\pi}(x) = \tilde{\pi}(x')$. The base change $U \times_X \operatorname{Spec} k = \operatorname{Spec}(A \otimes_{A^G} k)$ inherts a G-action and the G-orbits $Gx, Gx' \subseteq U \times_X A^G$ are closed subschemes. If $Gx \neq Gx'$, then the orbits are disjoint and there is a function $f \in A \otimes_R k$ with $f|_{Gx} = 0$ and $f|_{Gx'} = 1$. Then $\tilde{f} = \prod_{g \in G} f\, g \in (A \otimes_{A^G} k)^G$ is G-invariant function with $\tilde{f}(x) = 0 \neq \tilde{f}(x') = 1$. This implies $\tilde{\pi}(x) \neq \tilde{\pi}(x')$, a contradiction.

The map $\tilde{\pi} : U \to X$ is universal for G-invariant maps to algebraic spaces if and only if $\pi : \mathfrak{X} \to X$ is universal for maps to algebraic spaces. In other words, we need to show if Y is algebraic space, then the natural map

$$\operatorname{Mor}(X, Y) \to \operatorname{Mor}(\mathfrak{X}, Y) \qquad \text{(Eq. 4.7.1)}$$

is bijective. We note this is immediate when Y is affine as $\Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = \Gamma(X, \mathscr{O}_X)$ and the case when y is a scheme can be reduced to this case without much effort (if $g : \mathfrak{X} \to Y$ is a map, an affine covering Y_i of Y induces open covering $X_i = X \backslash \pi(\mathfrak{X} \backslash g^{-1}(Y_i))$ of X, and g restricts to $\pi^{-1}(X_i) \to Y_i$ which factors uniquely through X_i).

Now it remains to handle the case Y is algebraic space.

For injectivity, let $h_1, h_2 : X \to Y$ be two maps such that $h_1 \circ \pi = h_2 \circ \pi$. Let $E \to X$ be the equalizer of h_1 and h_2, i.e. the pullback of the diagonal $Y \to Y \times Y$ along $(h_1, h_2) : X \to Y \times Y$. The equalizer is a monomorphism and locally of finite type. By construction $\pi : \mathfrak{X} \to X$ factors through $E \to X$ and since π is universally closed and schematically dominant (i.e. $\mathscr{O}_X \to \pi_* \mathscr{O}_X$ is injective), so is $E \to X$. As every universally closed and locally of finite type monomorphism is a closed immersion, we conclude $E \to X$ is an isomorphism.

It remains to check surjectivity. We can show it suffices to check étale locally. Then, we can show after replacing X with étale cover $V \to X$ and $\mathfrak{X}$ with base change $\mathfrak{X} \times_X V$, there is a section $s : \mathfrak{X} \to \mathfrak{X}' := \mathfrak{X} \times_Y Y'$, where $Y' \to Y$ is an étale presentation, in the commutative diagram

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[r, "\pi"] \arrow[d, "g"] \arrow[l, bend right, "s"'] & X \arrow[dl, dashed] \\ Y' \arrow[r] & Y & \end{tikzcd}$$

The surjectivity follows from this claim: since X, Y' are affine, the equality $\Gamma(X, \mathscr{O}_X) = \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ implies $\mathfrak{X} \xrightarrow{s} \mathfrak{X}' \xrightarrow{g'} Y'$ factors through $\pi : \mathfrak{X} \to X$ via a morphism $X \to Y'$. The composition $X \to Y' \to Y$ yields the dotted arrow above, and hence we get surjectivity.

The proof of this claim is to reduce to the case of strict Henselian local rings via argument of limits, and we will not prove it.

Corollary 4.7.8.1

If G is a finite group acting freely on affine scheme $U = \operatorname{Spec} A$, then the algebraic space quotient U/G is isomorphic to $\operatorname{Spec} A^G$.

Now we will show a DM stack $\mathfrak{X}$ near a point x is étale locally the quotient stack $[\operatorname{Spec} A/G_x]$ of an affine scheme by stablizer group scheme. Conceptually, this tells us that just as schemes are obtained by gluing affine schemes in Zariski topology, DM stacks are obtained by gluing quotient stacks in étale topology. Note we also have algebraic spaces, which are obtained by gluing affine schemes in étale topology, so the level of complexity goes from schems to algebraic spaces to DM stacks, and then Artin stacks.

Let x be a point of DM stack $\mathfrak{X}$, then we define the geometric stablizer of x to be the stablizer group $G_{\overline{x}}$ for any geometric point $\operatorname{Spec} k \to \mathfrak{X}$ with image x.

Theorem 4.7.9: Local Structure Theorem

Let $\mathfrak{X}$ be a separated DM stack and $x \in \mathfrak{X}$ a point with geometric stablizer G_x. Then there is an affine étale morphism

$$f : ([\operatorname{Spec} A/G_x], w) \to (\mathfrak{X}, x)$$

where $w \in [\operatorname{Spec} A/G_x]$ such that f induces an isomorphism of geometric stablizer groups at w.

Proof. Choose a field-valued point $\operatorname{Spec} k \to \mathfrak{X}$ representing x. Let $(U, u) \to (\mathfrak{X}, x)$ be an étale representable morphism from an affine scheme, and let d be the degree over x, i.e. the cardinality of $\operatorname{Spec} k \times_{\mathfrak{X}} U$. Since $\mathfrak{X}$ is separated, $U \to \mathfrak{X}$ is affine. Define the affine scheme $(U/\mathfrak{X})^d = U \times_{\mathfrak{X}} U \ldots \times_{\mathfrak{X}} U$ where the product takes d times. For a scheme S, a morphism $S \to (U/\mathfrak{X})^d$ correspond to a morphism $S \to \mathfrak{X}$ and d sections $s_1, \ldots, s_d$ of $U_S := U \times_{\mathfrak{X}} S \to S$.

Let $(U/\mathfrak{X})_0^d$ be the quasi-affine subscheme of $(U/\mathfrak{X})^d$ which is the complement of all pairwise diagonals, i.e. a map $S \to (U/\mathfrak{X})_0^d$ corresponds to $S \to \mathfrak{X}$ and n sections $s_1, \ldots, s_d : S \to U_S$ which are disjoint (meaning the intersection of s_i, s_j is empty for $i \neq j$). There is an action of S_d on $(U/\mathfrak{X})^d$ by permuting the sections and $(U\mathfrak{X})_0^d$ is S_d-equivariant. By the correspondence between principal S_d-bundles and finite étale covers of degree d, an object of the quotient stack $[(U/\mathfrak{X})^d_0/S_d]$ over a scheme S corresponds to a diagram

$$\begin{tikzcd} Z \arrow[r, hook] & U_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ & S \arrow[r] & \mathfrak{X} \end{tikzcd}$$

where $Z \hookrightarrow U_S$ is a closed subscheme and $Z \to S$ is finite étale of degree d. Let $w \in [(U/\mathfrak{X})^d_0/S_d](k)$ be the point corresponding to $Z = \operatorname{Spec} k \times_{\mathfrak{X}} U$. There is an induced representable morphism $[(U/\mathfrak{X})^d_0/S_d] \to \mathfrak{X}$ and a commutative diagram

$$\begin{tikzcd} (U/\mathfrak{X})^d_0 \arrow[r, hook] \arrow[d] & (U/\mathfrak{X})^d \arrow[d] \\ {[(U/\mathfrak{X})^d_0/S_d]} \arrow[dr] & U \arrow[d] \\ & \mathfrak{X} \end{tikzcd}$$

Set $W = (U/\mathfrak{X})^d_0$. The morphism $[W/S_d] \to \mathfrak{X}$ is étale and representable, and induces an isomorphism of stablizer groups at w.

By quotienting out by $G_x \subseteq S_d$, the morphism $[W/G_x] \to \mathfrak{X}$ which is also étale and representable, and induces an isomorphism of stablizer groups at w. Let $W' \subseteq W$ be an affine open subscheme containing w, we may replace W with the G_x-invariant affine open subscheme $\bigcap_{g \in G_x} g \cdot W'$.

It remains to show $[W/G_x] \to \mathfrak{X}$ is affine. Since $\mathfrak{X}$ is separated, its diagonal is affine and the morphism $W \to \mathfrak{X}$ from the affine scheme W is affine. The fibered product

$$\begin{tikzcd} {[W/G_x] \times_{\mathfrak{X}} W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W \arrow[d] \\ {[W/G_x]} \arrow[r] & \mathfrak{X} \end{tikzcd}$$

is affine over $[W/G_x]$ and thus isomorphic to a quotient stack $[\operatorname{Spec} B/G_x]$. On the other hand, since $[W/G_x] \to \mathfrak{X}$ is representable, the quotient stack $[\operatorname{Spec} B/G_x]$ is an algebraic space and the action of G_x on $\operatorname{Spec} B$ is free. By Corollary 4.7.8.1, we see $[\operatorname{Spec} B/G_x]$ is isomorphic to $\operatorname{Spec} B^{G_x}$. By étale descent, we see $[W/G_x] \to \mathfrak{X}$ is affine.

Lemma 4.7.10

Let G be a finite group acting on affine scheme $\operatorname{Spec} A$. If $A^G \to B$ is flat ring homomorphism, then G acts on the affine scheme $\operatorname{Spec}(B \otimes_{A^G} A)$ and $B = (B \otimes_{A^G} A)^G$.

Lemma 4.7.11

Let $\pi : \mathfrak{X} \to X$ be a coarse moduli space such that for every étale morphism $X' \to X$

from an affine scheme, the base change $\mathfrak{X}\times_X X' \to X'$ is a coarse moduli space. Then the natural map $\mathscr{O}_X \to \pi_*\mathscr{O}_{\mathfrak{X}}$ is an isomorphism.

Proof. As π is universal for maps to algebraic spaces, we have $\mathrm{Mor}(X, \mathbb{A}^1) \to \mathrm{Mor}(\mathfrak{X}, \mathbb{A}^1)$ is bijective, i.e. in other words, $\Gamma(X, \mathscr{O}_X) \cong \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$. For every étale map $X' \to X$, the base c hange $\mathfrak{X}' = \mathfrak{X}\times_X X' \to X'$ is also a coarse moduli space, and thus $\Gamma(X', \mathscr{O}_{X'}) \cong \Gamma(\mathfrak{X}', \mathscr{O}_{\mathfrak{X}'})$. This shows $\mathscr{O}_X \to \pi_*\mathscr{O}_{\mathfrak{X}}$ is an isomorphism.

The next lemma says a given map is coarse moduli space can be checked étale locally.

Lemma 4.7.12

Let $\pi : \mathfrak{X} \to X$ be a morphism to an algebraic space. Suppose there is an étale covering $\{X_i \to X\}$ such that $\mathfrak{X}\times_X X_i \to X_i$ is a coarse moduli space for all i. Then $\pi : \mathfrak{X} \to X$ is a coarse moduli space.

Proof. The first condition follows from the fact that algebraic spaces are sheaves in the étale topology. The second condition is a condition on geometric fibers (i.e. fibers over field-valued points), thus can be checked étale locally.

Theorem 4.7.13

If G is a finite group acting on affine scheme $\mathrm{Spec}\,A$, then $\pi : [\mathrm{Spec}\,A/G] \to \mathrm{Spec}\,A^G$ is a coarse moduli space. Moreover,

1. *the base change of π along a flat morphism $X' \to \mathrm{Spec}\,A^G$ of algebraic spaces is a coarse moduli space*
2. *the natural map $X \to \pi_*\mathscr{O}_{\mathfrak{X}}$ is an isomorphism*
3. *if A is f.g. over Noetherian ring R, then A^G is f.g. over R and π is proper universal homeomorphism.*

Proof. We already seen $\pi : [\mathrm{Spec}\,A/G] \to \mathrm{Spec}\,A^G$ is coarse moduli space by Theorem 4.7.13.

To see (1), by Lemma 4.7.12 it suffices to consider flat morphism $Y' \to Y$ from an affine scheme. But in this case, the base change $\mathfrak{X}\times_Y Y'$ is isomorphic to a quotient stack $[\mathrm{Spec}\,B/G]$ and Lemma 4.7.10 implies $Y' \cong \mathrm{Spec}\,B^G$. It follows $\mathfrak{X}\times_Y Y' \to Y'$ is coarse moduli space.

Part (2) follows directly from (1) by Lemma 4.7.11.

For (3), we skip the commutative algebra statement. Since π is bijective and universally closed, its set-theoretic inverse is continuous, and thus π is a homeomorphism. The base change of π along a morphism $\operatorname{Spec} B \to \operatorname{Spec} A^G$ factors as

$$[\operatorname{Spec}(B \otimes_{A^G} A)/G] \to \operatorname{Spec}(B \otimes_{A^G} A)^G \to \operatorname{Spec} B$$

where the first map is a homeomorphism by the above argument and the second is a homeomorphism is left as an exercise. This shows π is universal homeomorphism.

Proposition 4.7.14

Let G be a finite group and $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a G-equivariant morphism of affine schemes of finite type over Noetherian R. Let $x \in \operatorname{Spec} A$ be a closed point. Assume f is étale at x and the induced map $G_x \to G_{f(x)}$ of stablizer group scheme is bijective. Then there exists open affine neighbourhood $W \subseteq \operatorname{Spec} A^G$ of the image of x such that $W \to \operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale and $\pi_A^{-1}(W) \cong W \times_{\operatorname{Spec} B^G} [\operatorname{Spec} B/G]$, where $\pi_A : [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$.

In other words, after replacing $\operatorname{Spec} A^G$ with ana ffine neighbourhood W of $\pi_A(x)$ and $\operatorname{Spec} A$ with $\pi_A^{-1}(W)$, it can be arranged that the diagram

$$\begin{tikzcd}
{[\operatorname{Spec} A/G]} \arrow[r, "f"] \arrow[d, "\pi_A"] & {[\operatorname{Spec} B/G]} \arrow[d, "\pi_B"] \\
\operatorname{Spec} A^G \arrow[r] & \operatorname{Spec} B^G
\end{tikzcd} \qquad \text{(Eq. 4.7.2)}$$

is Cartesian, where both horizontal maps are étale.

We also remark that the condition that $G_x \to G_{f(x)}$ is bijective can be checked on field-valued point $\operatorname{Spec} k \to \operatorname{Spec} A$ representing x, e.g. the inclusion of the residual field.

In the proof of Keel-Mori Theorem, the above proposition will be applied in the following form:

Corollary 4.7.14.1

Let G be a finite group and $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a G-equivariant morphism of affine schemes of finite type over a Noetherian ring R. Assume for every closed point $x \in \operatorname{Spec} A$, f is étale at x and the induced map $G_x \to G_{f(x)}$ is bijective. Then $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale and Eq. 4.7.2 is Cartesian.

Proof of Proposition 4.7.14. Set $y = f(x)$. We first claim the question is étale local around $\pi_B(y) \in \operatorname{Spec} B^G$. Indeed, if $Y' \to Y = \operatorname{Spec} B^G$ is an affine étale neighbourhood of $\pi_B(y)$, we let $X', \mathfrak{X}', \mathfrak{Y}'$ denote the base changes of $X = \operatorname{Spec} A^G$, $\mathfrak{X} := [\operatorname{Spec} A/G]$ and $\mathfrak{Y} := [\operatorname{Spec} B/G]$. By Lemma 4.7.10, we know $\mathfrak{Y}' \cong [\operatorname{Spec} B'/G]$ with $Y' \cong \operatorname{Spec} B'^G$ and similarly for $\mathfrak{X}'$ and X'. If the result holds after this base change, there is an open neighbourhood $W' \subseteq X'$ containing a preimage of $\pi_A(x)$ such that $W' \hookrightarrow X' \to Y'$ is étale and such that the preimage of W' in $\mathfrak{X}'$ is isomorphic to $W' \times_{Y'} \mathfrak{Y}'$. Taking W as the image of W' under $X' \to \operatorname{Spec} A^G$ and applying étale descent yields the desired claim.

We now claim this allows us to assume B^G is strictly Henselian. To see this, let $Y^{\mathrm{sh}} = \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{Y,\pi_B(y)}$ and X^{sh}, $\mathfrak{X}^{\mathrm{sh}}$ and $\mathfrak{Y}^{\mathrm{sh}}$ be the base changes of $X, \mathfrak{X}$ and $\mathfrak{Y}$ along $Y^{\mathrm{sh}} \to Y$. Suppose $U^{\mathrm{sh}} \to X^{\mathrm{sh}}$ is an open affine subscheme of the unique point in X^{sh} over x and the closed point of Y^{sh} such that $U^{\mathrm{sh}} \to Y^{\mathrm{sh}}$ is étale with $\pi^{-1}_{\mathfrak{X}^{\mathrm{sh}}}(U^{\mathrm{sh}}) \cong U^{\mathrm{sh}} \times_{Y^{\mathrm{sh}}} \mathfrak{Y}^{\mathrm{sh}}$. Then $Y = \lim_\lambda Y_\lambda$ is the limit of affine étale neighbourhood $Y_\lambda \to Y$ and we set X_λ, $\mathfrak{X}_\lambda$, and $\mathfrak{Y}_\lambda$ to be the base changes of $X, \mathfrak{X}$ and $\mathfrak{Y}$ along $Y_\lambda \to Y$. By standard limit argument, we see the morphism $U^{\mathrm{sh}} \to X^{\mathrm{sh}}$ descends to $U_\eta \to X_\eta$ for some η. Setting $U_\lambda = U_\eta \times_{X_\eta} X_\lambda$ for $\lambda > \eta$, it follows for $\lambda \gg 0$, we have $U_\lambda \to X_\lambda$ is an open immersion and the composition $U_\lambda \to X_\lambda \to Y_\lambda$ is étale and $\pi^{-1}_{\mathfrak{X}_\lambda}(U_\lambda) \cong U_\lambda \times_{Y_\lambda} \mathfrak{Y}_\lambda$.

Finally, as $B^G \to B$ is finite (Part (3) of Theorem 4.7.13), $B = B_1 \times \ldots \times B_r$ is a product of strictly Henselian local rings. As in the proof of Theorem 4.7.13, we may replace $[\operatorname{Spec} B/G]$ with $[\operatorname{Spec} B_1/G_y]$ and $[\operatorname{Spec} A/G]$ with $[f^{-1}(\operatorname{Spec} B_1)/G]$ to assume G fixes x and y while acting trivially on the residue field $\kappa(x) = \kappa(y)$. Thus $\operatorname{Spec} A \to \operatorname{Spec} B$ has a unique section $s : \operatorname{Spec} B \to \operatorname{Spec} A$ taking y to x. The section s is necessarily G-invariant (as in the proof of Theorem 4.7.13). Thus s descends to section of $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ which gives our desired open and closed subscheme $W \subseteq \operatorname{Spec} A^G$.

Remark 4.7.15

Here is a conceptual reason why we should expect this induced map of quotients to be étale. For simplicity, assume $R = k$ is algebraically closed field. Let $\hat{A}$ and $\hat{B}$ be completions of the local rings at x and $f(x)$. The stablizers G_x and $G_{f(x)}$ act on $\operatorname{Spec} \hat{A}$ and $\operatorname{Spec} \hat{B}$ respectively, and the map $\operatorname{Spec} \hat{A} \to \operatorname{Spec} \hat{B}$ is equivariant with respect to the map $G_x \to G_{f(x)}$. The completion $\widehat{A^G}$ of A^G at the image of x is isomorphic to $\widehat{A^{G_x}}$ and similarly $\widehat{B^G} = \widehat{B^{G_{f(x)}}}$. Since f is étale at x, $\hat{B} \to \hat{A}$ is an isomorphism and since $G_x \to G_{f(x)}$ is bijective, the induced map $\widehat{B^G} \to \widehat{A^G}$ is an isomorphism, which shows $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale at the image of x.

Now we are ready to prove the Keel-Mori Theorem.

Theorem 4.7.16

Let $\mathfrak{X}$ be a DM stack separated and of finite type over a Noetherian algebraic space S. Then there exists a coarse moduli space $\pi : \mathfrak{X} \to X$ with $\mathscr{O}_X = \pi_* \mathscr{O}_{\mathfrak{X}}$ such that:

1. *X is separated and of finite type over S,*
2. *π is proper universal homeomorphism, and*

> 3. *for every flat morphism $X' \to X$ of algebraic spaces, the base change $\mathfrak{X} \times_X X' \to X'$ is a coarse moduli space.*

There are other setting where this theorem holds, e.g. S be locally Noetherian, and $\mathfrak{X}/S$ have finite diagonal, or $I\mathfrak{X} \to \mathfrak{X}$ is finite.

Proof. We first handle the case when $S = \operatorname{Spec} R$ is affine. The question is Zariski-local on $\mathfrak{X}$: if $\{\mathfrak{X}_i\}$ is a Zariski open covering of $\mathfrak{X}$ with coarse moduli spaces $\mathfrak{X}_i \to X_i$, then since coarse moduli spaces are unique, the X_i's glue to form an algebraic space X and a map $\mathfrak{X} \to X$, which is a coarse moduli space by Lemma 4.7.12. It thus suffices to show every closed point $x \in |\mathfrak{X}|$ has an open neighbourhood which admits a coarse moduli space.

By the Local Structure Theorem 4.7.9, there is affine étale morphism

$$f : (\mathfrak{W} = [\operatorname{Spec} A/G_x], w) \to (\mathfrak{X}, x)$$

such that f induces an isomorphism of geometric stablizer groups at w.

We claim that since $\mathfrak{X}$ is separated, the locus $\mathfrak{U}$ consisting of points $z \in |\mathfrak{W}|$, such that f induces an isomorphism of geometric stablizer groups at z, is open. To establish this, we will analyze the natural morphism $I\mathfrak{W} \to I\mathfrak{X} \times_{\mathfrak{X}} \mathfrak{W}$ of relative group schemes over $\mathfrak{W}$ as the fiber of this morphism over $z \in \mathfrak{W}(k)$ is precisely the morphism $G_z \to G_{f(z)}$ of stablizers. We now exploit the Cartesian diagram (the proof is left as an exercise)

$$\begin{tikzcd} I\mathfrak{W} \arrow[r, "\Psi"] \arrow[d] \arrow[dr, phantom, "\square"] & I\mathfrak{X} \times_{\mathfrak{X}} \mathfrak{W} \arrow[d] \\ \mathfrak{W} \arrow[r] & \mathfrak{W} \times_{\mathfrak{X}} \mathfrak{W} \end{tikzcd}$$

Since $\mathfrak{W} \to \mathfrak{X}$ is representable, étale, and separated, the diagonal $\mathfrak{W} \to \mathfrak{W} \times_{\mathfrak{X}} \mathfrak{W}$ is an open and closed immersion and thus so is Ψ. Since $I\mathfrak{X} \to \mathfrak{X}$ is finite, so is $p_2 : I\mathfrak{X} \times_{\mathfrak{X}} \mathfrak{W} \to \mathfrak{W}$. Thus $p_2(|I\mathfrak{X} \times_{\mathfrak{X}} \mathfrak{W}| \setminus |I\mathfrak{W}|) \subseteq |\mathfrak{W}|$ is closed and its complement, which is identified with the locus $\mathfrak{U}$, is open.

Let $\pi_{\mathfrak{W}} : \mathfrak{W} \to W = \operatorname{Spec} A^{G_x}$ be the coarse moduli space (Theorem 4.7.13). Choose an affine open subscheme $X_1 \subseteq W$ containing $\pi_{\mathfrak{W}}(w)$. Then $\mathfrak{X}_1 = \pi_{\mathfrak{W}}^{-1}(X_1)$ is isomorphic to a quotient stack $[\operatorname{Spec} A_1/G_x]$ such that $X_1 = \operatorname{Spec} A_1^{G_x}$. This provides an affine étale morphism

$$g : (\mathfrak{X}_1 = [\operatorname{Spec} A_1/G_x], w) \to (\mathfrak{X}, x)$$

which induces a bijection on all geometric stablizer groups.

We now show the open substack $\mathfrak{X}_0 = \operatorname{im}(f)$ admits a coarse moduli space. Define $\mathfrak{X}_2 = \mathfrak{X}_1 \times_{\mathfrak{X}} \mathfrak{X}_1$ and $\mathfrak{X}_3 = \mathfrak{X}_1 \times_{\mathfrak{X}} \mathfrak{X}_1 \times_{\mathfrak{X}} \mathfrak{X}_1$. Since g is affine, each $\mathfrak{X}_i$ is of the form $[\operatorname{Spec} A_i/G_x]$ and there is a coarse moduli space $\pi_i : \mathfrak{X}_i \to X_i = \operatorname{Spec} A_i^{G_x}$. By universality of coarse moduli spaces, there is a diagram

$$\begin{tikzcd} \mathfrak{X}_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & \mathfrak{X}_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathfrak{X}_1 \arrow[r] \arrow[d] & \mathfrak{X}_0 = \operatorname{im}(f) \arrow[d] \\ X_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_2 \arrow[r, shift left] \arrow[r, shift right] & X_1 \arrow[r, dashed] & X_0 \end{tikzcd} \tag{Eq. 4.7.3}$$

where the natural squares commute. Since g induces bijection of geometric stablizer groups at all points, the same is true for each projection $\mathfrak{X}_2 \to \mathfrak{X}_1$ and $\mathfrak{X}_3 \to \mathfrak{X}_2$. Corollary 4.7.14.1 implies that each map $X_2 \to X_1$ and $X_3 \to X_2$ is étale, and the natural square of solid arrows in Eq. 4.7.3 are Cartesian.

The universality of coarse moduli spaces induces an étale groupoid structure $X_2 \rightrightarrows X_1$. To check this is an étale equivalence relation, it suffices to check $X_2 \to X_1 \times X_1$ is injective on geometric points but this follows from the observation that $|\mathfrak{X}_2| \to |\mathfrak{X}_1| \times |\mathfrak{X}_1|$ is injective on closed points. Therefore there is an algebraic space quotient $X_0 := X_1/X_2$ and a map $X_1 \to X_0$. By étale descent along $\mathfrak{X}_1 \to \mathfrak{X}_0$, there is a map $\pi_0 : \mathfrak{X}_0 \to X_0$ making the right square in Eq. 4.7.3 commute.

To argue $\pi : \mathfrak{X}_0 \to X_0$ is a coarse moduli space, we will use the commutative cube

$$\begin{tikzcd}
 & \mathfrak{X}_2 \arrow[rr] \arrow[dl] \arrow[dd] & & \mathfrak{X}_1 \arrow[dl] \arrow[dd] \\
\mathfrak{X}_1 \arrow[rr, crossing over] \arrow[dd] & & \mathfrak{X}_0 & \\
 & X_2 \arrow[rr] \arrow[dl] & & X_1 \arrow[dl] \\
X_1 \arrow[rr] & & X_0 \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the top, left and bottom faces are Cartesian. It follows from étale descent along $\mathfrak{X}_1 \to \mathfrak{X}_0$ that the right face is also Cartesian and since being a coarse moduli space is étale local on X_0 (Lemma 4.7.12), we conclude $\mathfrak{X}_0 \to X_0$ is coarse moduli space. Except for the separatedness, the additional properties in the statement are étale-local on X_0 so they follows from the analogous properties in Theorem 4.7.13. As $\mathfrak{X}_0 \to X_0$ is proper, the separatedness of $\mathfrak{X}_0$ is equivalent to separatedness of X_0.

Finally,the case when when S is a Noetherian algebraic space can be reduced to the affine case by imitating the above argument to étale locally construct the coarse moduli space of $\mathfrak{X}$.

Corollary 4.7.16.1: Local Structure of Coarse Moduli Spaces

Let $\mathfrak{X}$ be DM stack of finite type and separated over Noetherian algebraic space S, and $\pi : \mathfrak{X} \to X$ its coarse moduli space. For every closed point $x \in |\mathfrak{X}|$ with geometric stablizer group G_x, there is a Cartesian diagram

$$\begin{tikzcd}
{[\mathrm{Spec} A/G_x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "\pi"] \\
\mathrm{Spec} A^{G_x} \arrow[r] & X
\end{tikzcd}$$

such that $\operatorname{Spec} A^{G_x} \to X$ *is an étale neighbourhood of* $\pi(x) \in |X|$.

Example 4.7.17

Consider the moduli stack $M_{1,1}$ of elliptic curves over a field k with $\operatorname{char}(k) \neq 2, 3$. The Weierstrass form $y^2 = x(x-1)(x-\lambda)$ gives an isomorphism $M_{1,1} \cong [(\mathbb{A}^1 \backslash \{0,1\})/S_3]$ where the S_3-orbit of λ is

$$\{\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{1-\lambda}, \frac{\lambda-1}{\lambda}\}$$

The coarse moduli space is given by j-invariant

$$j : M_{1,1} \to \mathbb{A}^1, \quad \lambda \mapsto 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^3}$$

Indeed

To conclude this section, we mention a generalization of the local structure theorem for DM stacks to algebraic stacks.

Theorem 4.7.18: Local Structure Theorem for Algebraic Stack

Let $\mathfrak{X}$ *be an algebraic stack of finite type over an AFC* k *with affine diagonal. For every point* $x \in \mathfrak{X}(k)$ *with linearly reductive stablizer* G_x*, there is an affine étale morphism*

$$f : ([\operatorname{Spec} A/G_x], w) \to (\mathfrak{X}, x)$$

which induces an isomorphism of stablizer groups at w.

4.8 Good Moduli Space

We note KM (Keel-Mori) theorem only handles DM stacks in char 0. What about Artin stacks?

Typically, Artin stacks don't have corase space, e.g. $[\mathbb{A}^1/\mathbb{G}_m]$. Heuristically, this is because Artin stacks are rarely separated (and if we have corase space, it implies its proper and hence separated).

Jarod Alper in 2009 or 2010 introduced a notion of corase space for Artin stacks: they are called good moduli spaces (gms). We will talk about the idea.

Definition 4.8.1

We say a map $f : \mathfrak{X} \to \mathfrak{Y}$ is ***cohomologically affine*** if $f_* : (\mathbf{Qcoh})(\mathfrak{X}) \to (\mathbf{Qcoh})(\mathfrak{Y})$ is exact.

Theorem 4.8.2: Serre

If $f : \mathfrak{X} \to \mathfrak{Y}$ is a representable map, then its cohomologically affine if and only if it is affine.

Definition 4.8.3

We say $f : \mathfrak{X} \to \mathfrak{Y}$ is ***Stein*** if $f_* \mathscr{O}_{\mathfrak{X}} = \mathscr{O}_{\mathfrak{Y}}$.

Example 4.8.4

1. KM theorem says if $\mathfrak{X}$ has finite diagonal, then coarse map is Stein.
2. If X is normal (e.g. smooth) scheme and $U \subseteq X$ open with $\operatorname{codim}(X \setminus U) \geq 2$ then the inclusion map $i : U \to X$ is Stein.
3. If $f : X \to Y$ is proper with connected fibers, e.g. any projective variety $X \to \operatorname{Spec} k$, then f is Stein.

Remark 4.8.5

If $f : X \to Y$ with X, Y schemes, and f is affine and Stein, then f is an isomorphism. More generally, this is true if $f : \mathfrak{X} \to \mathfrak{Y}$ is representable.

Definition 4.8.6

Let $\pi : \mathfrak{X} \to X$ with X algebraic space. Then we say π is ***good moduli space*** if π is cohomologically affine and Stein.

Example 4.8.7

Let G be an affine algebraic group over k. Then a coarse space map $\pi : BG \to \operatorname{Spec} k$ is Stein. Hence it is gms iff it is cohomologically affine. In particular, note $\pi_* : (\mathbf{Qcoh})(BG) \to (\mathbf{Qcoh})(\operatorname{Spec} k)$ correspond to a map from k-vector spaces with G-action to k-vector spaces, i.e. π_* is given by $G \circlearrowleft V \mapsto V^G$. By definition, π is cohomologically affine iff taking G-invariants is exact.

These are called ***linearly reductive groups***.

Example 4.8.8

Let G be finite group, then G is linearly reductive iff $\operatorname{char} k \nmid |G|$. Recall when we prove every representation of G can be decompose into irreducible, we did something like $0 \to V' \to V \to V'' \to 0$ and we need to find a G-equivariant splitting. To do this, you need to divide by $|G|$ (hence we want $\operatorname{char} k \nmid |G|$).

Example 4.8.9

The following groups are all linearly reductive: SL_n, GL_n, Sp_n, SO_n, tori. A non-example would be $\mathbb{G}_a = \mathbb{A}^1$ considered as a group under addition. To see this, consider $\mathbb{G}_a \to \mathrm{GL}_2$ via the map

$$x \mapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

this correspond to a sequence $0 \to k \to k^{\oplus 2} \to k \to 0$ but the map is not diagonalizable and hence it does not split, i.e. $\mathbb{G}_a$ is not linearly reductive.

Example 4.8.10

Let G be linearly reductive group scheme and let G acts on $\operatorname{Spec} A$. Then $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is gms map. For example, $[\mathbb{A}^1/\mathbb{G}_m] \to \operatorname{Spec} k[x]^{\mathbb{G}_m} = \operatorname{Spec} k$ is a gms. Thus, we see the notion of gms is sort of containing the notion of corase space, but that's not always the case.

We also note, this example shows that if we have gms $\pi : \mathfrak{X} \to X$, the points of $\mathfrak{X}(k)$ up to isomorphism are not in bijection with $X(k)$.

Example 4.8.11

Consider the corase map $B\mathbb{G}_a \to \operatorname{Spec} k$. However, this is not gms.

Theorem 4.8.12: Alper

1. *gms are universal maps to algebraic spaces if the base is locally Noetherian.*
2. *gms commutes with arbitrary base change.*
3. *let $\pi : \mathfrak{X} \to X$ be gms, then for all $x : \operatorname{Spec} k \to X$ with $k = \overline{k}$, there exists unique closed point in $\mathfrak{X}$ above x.*

There is now a version of KM theorem for gms, i.e. for Artin stacks. This is a result by Alper, Halpern-Leistner, Heinloth in 2019.

Remark 4.8.13

If $\mathfrak{X}$ has finite diagonal and $\mathfrak{X}$ is DM, then the corase space $\pi : \mathfrak{X} \to X$ is a good moduli space if and only if all stablizer groups are prime to characteristic.

Theorem 4.8.14

Let S be locally Noetherian, $\mathfrak{X}/S$ be DM with finite diagonal. Let $\pi : \mathfrak{X} \to X$ be coarse space. Let $\overline{x} : \operatorname{Spec} k \to X$ with $k = \overline{k}$. Since π is coarse, there exists a unique lift $\tilde{x} \in \mathfrak{X}(k)$. Let $G_{\tilde{x}}$ be the automorphism group, which is finite because $\Delta_{\mathfrak{X}/S}$ is finite.

Then, there exists etale neighbourhood $U \xrightarrow{et} X$, such that the pullback is given

> *by*
>
> $$\begin{tikzcd} {[V/G_{\tilde{x}}]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ U \arrow[r, "et"] & X \end{tikzcd}$$
>
> *with $V \to U$ finite and $G_{\tilde{x}} \circlearrowright V$.*

Proof. We give a sketch proof.

It is enough to prove this where U is the "etale stalk of X". The etale stalk is $\operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,x}$ where sh stands for strict Henselization (it is sort of like a completion/extension that makes Hensel's lemma holds).

Then, there exists etale $V \twoheadrightarrow \mathfrak{X}$ such that the sequence $V \to \mathfrak{X} \to X$ is quasi-finite in neighbourhood of x. Now take pullback, we get

$$\begin{tikzcd} V_{\overline{x}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ \mathfrak{X}_{\overline{x}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\overline{x}} \arrow[r] & X \end{tikzcd}$$

Now we import a black box statement without proof: $V_{\overline{x}} \to \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\overline{x}}$ is quasi-finite, so $V_{\overline{x}} = W_1 \coprod W_2$ where $W_1 \to \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\overline{x}}$ is finite and $W_2 \to \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\overline{x}}$ misses $\overline{x}$.

Let V' be one connected component of W_1. Then we have $V' \subseteq V_{\overline{x}} \twoheadrightarrow \mathfrak{X}_{\overline{x}}$ where we note $V' \twoheadrightarrow \mathfrak{X}_{\overline{x}}$ is etale cover because it hits closed point. Thus we get the following sequence

$$\begin{tikzcd} \mathfrak{Z}' = V' \times_{\mathfrak{X}_{\overline{x}}} V' \arrow[r, shift left, "et"] \arrow[r, shift right, "et"'] & V' \arrow[r, "et"'] \arrow[rr, bend left, "finite"] & \mathfrak{X}_{\overline{x}} \arrow[r] & \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\overline{x}} \end{tikzcd}$$

and it becomes

$$\begin{tikzcd} \mathfrak{Z}' = V' \times_{\mathfrak{X}_{\overline{x}}} V' \arrow[r, shift left, "et"] \arrow[r, shift right, "et"'] & V' \arrow[r, "et"'] \arrow[rr, bend left, "finite"] & \mathfrak{X}_{\overline{x}} \arrow[r] & \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\overline{x}} \\ G_{\tilde{x}} \times V'_k \arrow[u] \arrow[r, shift left] \arrow[r, shift right] \arrow[ur, phantom, "\square"] & V'_k \arrow[r] & \mathfrak{X}_k \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \operatorname{Spec} k \arrow[u] \end{tikzcd}$$

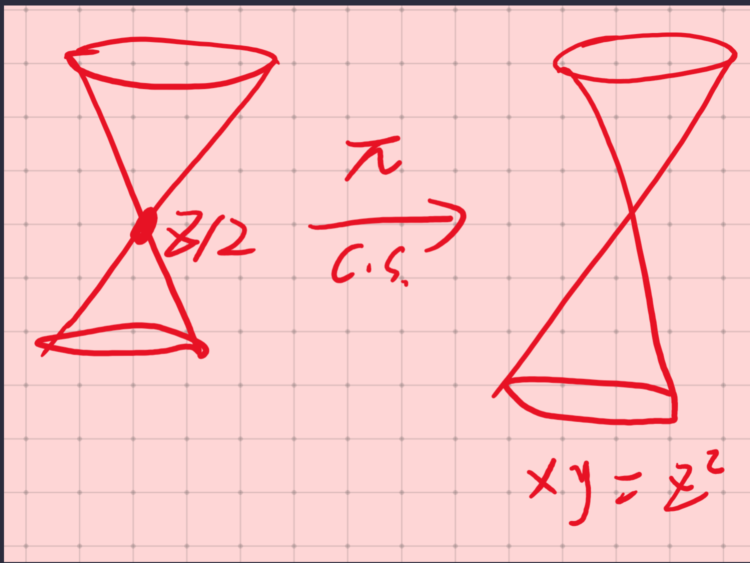
By deformation theory, there exists a deformation of $G_{\tilde{x}} \times V'_k \to V'_k$ over V', therefore, $\mathfrak{Z}' \cong V' \times G_{\tilde{x}} \to V'$ (this is called invariance of the etale site).

However, note we have $G_{\tilde{x}} \times V' \rightrightarrows V'$ where one arrow is the projection, and the second arrow is σ. What is σ?

The groupoid structure on $\mathfrak{X}_{\overline{x}}$ turns σ into a group action map, i.e. $\mathfrak{X}_{\overline{x}} = [V'/G_{\tilde{x}}]$

Example 4.8.15

Recall we had example $[\mathbb{A}^2/\mathbb{Z}/2]$ which looks like a cone



We know π is proper, coarse space map. If we consider the pullback

$$\begin{tikzcd} {} \arrow[r, "\subseteq"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} = \text{smooth} \arrow[d] \\ X^{sm} \arrow[r] & X \end{tikzcd}$$

So π is birational, proper map from a smooth stack. So π is a "stacky resolution".

Now, let's ask what can we say about X by looking at $\mathfrak{X}$. Hodge theory says that over $\mathbb{C}$, we have decompostion aobut $H^*(Y)$ with Y projective over $\mathbb{C}$, i.e. $H^n(Y)$ breaks up into finer invariants isomorphic to $\bigoplus_{p+q=n} H^q(\Omega^p_{Y/\mathbb{C}})$. This is called Hodge decomposition.

Mirror symmetry is about: we start with Calabi-Yau Y, and we can show there exists a mirrow Y^* where the $\dim H^q(\Omega^p_{Y^*})$ are $\dim H^q(\Omega^p_Y)$ up to a "flip".

Now, in our example, we have a variety X with $\mathfrak{X}$ lying above. It turns out, we can develop a version of Hodge theory of $\mathfrak{X}$ and it gives the Hodge theory of X (this is the work of Steenbrink, where he called those things V-manifolds).

Question: is there a smooth DM stack $\mathfrak{X} \xrightarrow[c.s.]{\pi} \operatorname{Spec} k[x,y,z]/(xy - z^2)$ in char 2?

The answer is no. However, there exists a smooth Artin stack $\mathfrak{X}$ with finite Δ with X as coarse space.

It is given by $[\mathbb{A}^2/\mu_2]$ where μ_2 is 2nd roots of unity, where in char(2) it is different from $\mathbb{Z}/2$, i.e. $\#\{\mu_2(\mathbb{F}_2[x]/x^2)\} \neq 2$. In particular, this is a singular group scheme and it is linearly reductive.

4.9 Canonical And Root Stacks

In this section we consider two constructions, called canonical stacks and root stacks.

If G is a finite group with $\operatorname{char} k \nmid |G|$ and if $G \circlearrowleft \mathbb{A}_k^n$ is a linear representation, then there is a characterization of when $\mathbb{A}_k^n/G$ is smooth.

Example 4.9.1

Let $\mathbb{Z}/2$ acts on $\mathbb{A}_k^2$ via $(x,y) \mapsto (x,-y)$. Then

$$\mathbb{A}_k^2/(\mathbb{Z}/2) = \operatorname{Spec} k[x,y]^{\mathbb{Z}/2} = \operatorname{Spec} k[x,y^2] \cong \mathbb{A}_k^2$$

This is smooth.

On the other hand, let $\mathbb{Z}/2$ acts on $\mathbb{A}_k^2$ via $(x,y) \mapsto (-x,-y)$. Then we get

$$\mathbb{A}_k^2/(\mathbb{Z}/2) = \operatorname{Spec} k[x^2, xy, y^2] = \operatorname{Spec} k[a,b,c]/(ac-b^2)$$

This is singular.

Let's compare this two examples. In example 1, the fixed locus of the action is $y=0$, a hyperplane. In example 2, the fixed locus is $x=y=0$.

Definition 4.9.2

We say $1 \neq g \in G$ is a ***pseudo-reflection*** if its fixed locus is a hyperplane, i.e. if g has all but one eigenvalue 1.

Note reflection means all but one eigenvalue equal 1 and the last eigenvalue equal -1. Then pseudo-reflection relaxs the last condition, i.e. we allow the last eigenvalue to be some root of unity.

Theorem 4.9.3: Chevalley–Shephard–Todd

Let G be a finite group, $G \circlearrowleft \mathbb{A}_k^n$ with $\operatorname{char} k \nmid |G|$. Then $\mathbb{A}_k^n/G \cong \mathbb{A}_k^n$ iff $\mathbb{A}_k^n/G$ is smooth iff the subgroup $H \subseteq G$ generated by pseudo-reflections is all of G.

What this means is if $X = \mathbb{A}_k^n/G$ for some finite group G with $\operatorname{char} k \nmid |G|$, then there exists a minimal way to write X in this form.

For example, $\mathbb{A}_k^2 \cong \mathbb{A}_k^2/(\mathbb{Z}/2)$ as in the first example. But there was a more minimal way to write $\mathbb{A}_k^2$ as a quotient by a finite group, namely $\mathbb{A}_k^2 \cong \mathbb{A}_k^2/\text{trivial group}$.

Definition 4.9.4

A variety X/k has ***quotient singularity*** if there exists a etale cover $\{X_i \xrightarrow{et} X\}$ such that $X_i = V_i/G_i$ with V_i smooth over k and G_i finite.

In the above we proved if $\mathfrak{X}$ is smooth DM, then its coarse space X has quotient singularity since we showed

$$\begin{tikzcd} {[V/G]} \cong \mathfrak{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "\pi"] \\ X' \arrow[r, two heads, "et"] & X \end{tikzcd}$$

with V smooth and G finite.

Conversely, in Vistoli's thesis, he showed the following

Theorem 4.9.5: Vistoli

For all X with quotient singularity prime to the characteristic, there exists a canonical smooth DM stack $\mathfrak{X}$ with coarse space X. Furthermore, $\mathfrak{X} \xrightarrow{\pi} X$ is an isomorphism over X^{sm} the smooth locus of X.

Vistoli proves this by looking locally where $X_i = V_i/G_i$ and G has no pseudo-reflections. Then he lets $\mathfrak{X}_i = [V_i/G_i]$ and proves the $\mathfrak{X}_i$ glues.

This is called the ***canonical stack*** of X, we denote it by X^{can}.

Next, we consider the root stacks.

Let X be a scheme, a techinical problem one often encounters in algebraic geometry concerns taking roots of divisors. Specifically, given an integer n and an effective Cartier divisor $D \subseteq X$, it is sometimes of interest to find another effective Cartier divisor E such that $nE = D$. This is of course not possible in general, and the best we can hope for is to find $f : Y \to X$ so f^*D makes sense on Y and such that there is E on Y so $nE = f^*D$. The root stack construction is an attempt at finding a universal such (Y, E).

Definition 4.9.6

Let X be a scheme. A ***generalized effective Cartier divisor*** on X is a pair (L, ρ) where L is a line bundle and $\rho : L \to \mathscr{O}_X$ is a morphism of $\mathscr{O}_X$-modules. An isomorphism between two generalized effective divisors (L, ρ) and (L, ρ') is an isomorphism $\sigma : L' \to L$ with commutative diagram

$$\begin{tikzcd} L' \arrow[rr, "\sigma"] \arrow[dr, "\rho'"'] & & L \arrow[dl, "\rho"] \\ & \mathscr{O}_X & \end{tikzcd}$$

Example 4.9.7

Let $D \subseteq X$ be an effective Cartier divisor, then the ideal sheaf $\mathscr{I}_D$ together with the inclusion $j_D : \mathscr{I}_D \to \mathscr{O}_X$ is a generalized effective Cartier divisor on X. Note if $D, D' \subseteq X$ are two effective Cartier divisors, then $(\mathscr{I}_D, j_D)$ and $(\mathscr{I}_{D'}, j_{D'})$ are isomorphic if and only if $D = D'$, in which case the isomorphism is unique.

If (L, ρ) and (L', ρ') are two generalized effective Cartier divisors, then we can form the product $(L, \rho) \cdot (L', \rho') = (L \otimes L', \rho \otimes \rho')$, where we write $\rho \otimes \rho'$ for the map

$$L \otimes L' \xrightarrow{\rho \otimes \rho'} \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{O}_X \xrightarrow{\sim} \mathscr{O}_X$$

Thus, we can define $(L^{\otimes n}, \rho^{\otimes n})$ as the n-fold product of (L, ρ) itself.

Now let $\mathrm{Div}^+(X)$ denote the isomorphism classes of generalized effective Cartier divisors, and with the product above, we obtain a commutative monoid with unit.

Remark 4.9.8

The main advantage of working with generalized effective Cartier divisors is that they pullback easily. Namely, if $g : Y \to X$ is a morphism of schemes and $(L, \rho) \in \mathrm{Div}^+(X)$, then g^*L with the map $g^*\rho : g^*L \to g^*\mathscr{O}_X = \mathscr{O}_Y$ is a generalized effective Cartier divisor on Y.

Now consider the fibered category $\mathfrak{D}$ over the category of schemes, whose objects are pairs $(T, (L, \rho))$, where T is a scheme and $(L, \rho) \in \mathrm{Div}^+(T)$. A morphism $(T', (L', \rho')) \to (T, (L, \rho))$ is a pair $(g, g^\flat)$ consisting of a morphism $g : T' \to T$ and an isomorphism $g^\flat : (L', \rho') \to (g^*L, g^*\rho)$.

Note that since invertible sheaves and morphisms between them satisfy effective descent, the fibered category $\mathfrak{D}$ is a stack.

Proposition 4.9.9

The stack $\mathfrak{D}$ is isomorphic to the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ of $\mathbb{A}^1$ by standard multiplication action of $\mathbb{G}_m$. In particular, $\mathfrak{D}$ is an algebraic stack.

Proof. Consider the fibered category $\{\mathbb{A}^1/\mathbb{G}_m\}$ whose objects are pairs (T, f) where T a scheme and $f \in \Gamma(T, \mathscr{O}_T)$. The morphisms from (T', f') to (T, f) is given by (g, u), where $g : T' \to T$ and $u : \Gamma(T', \mathscr{O}_{T'}^*)$ is a unit such that $f' = u \cdot g^\sharp(f) \in \Gamma(T', \mathscr{O}_{T'})$. We have a morphism of fibered categories

$$\{\mathbb{A}^1/\mathbb{G}_m\} \to \mathfrak{D}$$

sending (T, f) to $(T, (\mathscr{O}_T, \cdot f))$ and a morphism (g, u) to the morphism

$$(T', (\mathscr{O}_{T'}, \cdot f')) \to (T, (\mathscr{O}_T, \cdot f))$$

given by g and multiplication by u on $\mathscr{O}_{T'}$. Note this morphism is fully faithful and every object of $\mathfrak{D}$ is locally in the image. Thus this morphism of fibered categories induces an isomorphism $[\mathbb{A}^1/\mathbb{G}_m] \to \mathfrak{D}$.

Q.E.D.

The morphisms $\mathbb{A}^1 \to \mathbb{A}^1$ and $\mathbb{G}_m \to \mathbb{G}_m$ given by $t \mapsto t^n$ define a morphism of stacks

$$p_n : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$$

Under the identification of Proposition 4.9.9, this morphism p_n correspond to the morphism of stacks

$$\begin{aligned} \mathfrak{D} &\to \mathfrak{D} \\ (T,(L,\rho)) &\mapsto (T,(L^{\otimes n},\rho^{\otimes n})) \end{aligned}$$

Now fix a generalized Cartier divisor (L,ρ) and an integer $n \geq 1$. Let $\mathfrak{X}_n = \sqrt[n]{X/L}$ be the fibered category over the category of schemes, whose objects are triples $(f : T \to X, (M,\lambda), \sigma)$, where $f : T \to X$ is an X-scheme, (M,λ) is a generalized effective divisor on T, and

$$\sigma : (M^{\otimes n}, \lambda) \to (f^*L, f^*\rho)$$

is an isomorphism of generalized effective Cartier divisors on T. A morphism

$$(f' : T' \to X, (M',\lambda'), \sigma') \to (f : T \to X, (M,\lambda), \sigma)$$

is a pair $(h, h^\flat)$ where $h : T' \to T$ is an X-morphism, and $h^\flat : (M',\lambda') \to (h^*M, h^*\lambda)$ is an isomorphism of generalized effective Cartier divisors on T' such that the diagram

$$\begin{tikzcd} M'^{\otimes n} \arrow[rr, "(h^\flat)^{\otimes n}"] \arrow[dr, "\lambda'"'] & & h^*M^{\otimes n} \arrow[dl, "h^*\lambda"] \\ & f'^*L \cong h^*f^*L & \end{tikzcd}$$

commutes.

Definition 4.9.10

This fibered category $\mathfrak{X}_n = \sqrt[n]{X/L}$ over $(\mathbf{Sch}/X)$ is called the ***nth root stack*** associated to (L,ρ).

Note the projection $\pi_n : \mathfrak{X}_n \to (\mathbf{Sch}/X)$ is given by sending $(f : T \to X, (M,\rho), \sigma)$ to $f : T \to X$.

Theorem 4.9.11

1. *The fibered category $\mathfrak{X}_n$ is an algebraic stack with finite diagonal.*
2. *If $L = \mathcal{O}_X$ and ρ is given by an element $f \in \Gamma(X, \mathcal{O}_X)$, then $\mathfrak{X}_n$ is isomorphic to the quotient stack*
$$\mathrm{Spec}_X(\mathcal{O}_X[T]/(T^n - f))$$
by the action of μ_n given by $\xi \cdot T = \xi T$.
3. *The map π_n is an isomorphism over the open subset $U \subseteq X$ where ρ is an isomorphism.*
4. *If n is invertible in X, then $\mathfrak{X}_n$ is a DM stack.*

We see that root stacks are a way of putting stacky structure in codimension 1 although it only adds Abelian stablizers.

It turns out that every smooth DM stack with trivial generic stablizer can be built out of these two operations: canonical stacks and root stacks.

Theorem 4.9.12: The Bottom-Up Theorem, Geraschenko & Satriano

If $\mathfrak{X}$ is a smooth separated DM stack with trivial generic stablizer and if X is coarse space of $\mathfrak{X}$. Then $\mathfrak{X}$ is a canonical stack over a root stack over the canonical stack of X.

More specifically, if $\pi : \mathfrak{X} \to X$ has ramification divisor $D = D_1 \cup D_2 \ldots \cup D_n \subseteq X$ and π is ramified to order e_i along D_i, then $\mathfrak{R} = (\sqrt[e]{X^{\mathrm{can}}/D})^{\mathrm{can}}$ where $\sqrt[e]{X^{\mathrm{can}}/D}$ means take the e_ith root stack along D_i.