When Is Linear Recursion Non-negative

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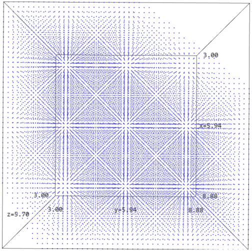

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What about $u_3 = (2\sqrt{5}+3)u_2 - 3(2\sqrt{5}+3)u_1 - 27u_0$?

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We say $\mathbf{u}$ is:

1. ***positive*** if $u_n \geq 0$ for all $n \geq 0$
2. ***ultimately positive*** if $\exists N$ so $u_n \geq 0$ for all $n \geq N$

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3. question 3 and 4 are still open for large depth

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We want to know whether those sequences will be ultimately positive or not because some models would not have real-life meaning if the values are negative

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2. The ***dominant roots*** of $\mathbf{u}$ are the roots with maximum modulus.

Theorem

Theorem 2

Let **u** *be a LRS, then*

$$u_n = p_1(n)\gamma_1^n + \ldots + p_k(n)\gamma_k^n$$

where $p_i(x) \in \mathbb{C}[x]$ *are polynomials, and* γ_i *are the characteristic roots.*

Proof: Simple Case

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$$C_{\mathbf{u}} := \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \dots & a_{k-1} & a_k \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

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Then we see

$$\begin{bmatrix} u_{n+k} \\ u_{n+k-1} \\ \vdots \\ u_{n+1} \end{bmatrix} = C_{\mathbf{u}} \begin{bmatrix} u_{n+k-1} \\ \vdots \\ u_n \end{bmatrix}$$

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Hence the power of $C_{\mathbf{u}}$ is very easy to compute, and thus

$$\begin{bmatrix} u_n \\ u_{n-1} \\ \vdots \\ u_{n-k+1} \end{bmatrix} = PD^nP^{-1} \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$$

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Remark

Since P, D are all matrices, we see in this case p_i in our theorem are all constant polynomials.

Example

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The diagonalization is given by

$$C_F = \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{5}} & -\frac{1-\sqrt{5}}{4\sqrt{5}} \\ -\frac{1}{2\sqrt{5}} & \frac{1+\sqrt{5}}{4\sqrt{5}} \end{bmatrix}$$

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Hence

$$F_n = \frac{1}{\sqrt{5}}\phi_2^n - \frac{1}{\sqrt{5}}\phi_1^n$$

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1. Set $U(x) = \sum_{n \geq 0} u_n x^n$
2. Assume $n \geq k$, then we see

$$\begin{cases} U(x) = \dots + u_n x^n \\ xU(x) = \dots + u_{n-1} x^n \\ x^2 U(x) = \dots + u_{n-2} x^n \\ \vdots \end{cases}$$

Proof: General Case

3. Thus we see

$$(1 - a_1 x - a_1 x^2 - \ldots - a_k x^k) U(x) = \ldots + (u_n - a_1 u_{n-1} - \ldots) x^n + \ldots$$

4. In other word, the RHS cannot contain terms with degree higher than $k-1$. Denote this by $G(x)$.
5. Thus

$$(x^k f_{\mathbf{u}}(1/x)) U(x) = G(x) \Rightarrow U(x) = \frac{G(x)}{x^k f_{\mathbf{u}}(1/x)}$$

6. Therefore, u_n is exactly the nth coefficient of the Taylor expansion of $G(x)/(x^k f_{\mathbf{u}}(1/x))$ around $x = 0$, but one can verify that this gives the desired closed form

Examples

Example

1. If $u_n = 2u_{n-1} - u_{n-2}$ and $u_0 = 3$, $u_1 = 1$, then $f_{\mathbf{u}}(x) = (x-1)^2$ and

$$u_n = p_1(n) \cdot 1 + p_2(n) \cdot 1, \quad p_1(n) = 3, p_2(n) = -2n$$

2. If $u_n = 2u_{n-1} + u_{n-2}$ with $(u_0, u_1) = (3, 1)$ then

$$u_n = \frac{1}{2}((3+\sqrt{2})\gamma_1^n - (\sqrt{2}-3)\gamma_2^n)$$

with $\gamma_1 = 1 - \sqrt{2}$ and $\gamma_2 = 1 + \sqrt{2}$.

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Next, we will try to give a basic idea of how to prove a statement like this

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2. α is ***algebraic integer*** if p_α is monic polynomial
3. the collection of algebraic integers is denoted by $\mathcal{O}$

Number Fields

Definition 8

A ***number field*** K is a field K such that $\mathbb{Q} \subseteq K$ and K is finite dimensional vector space over $\mathbb{Q}$. The ***ring of integers*** for K is defined by $\mathcal{O}_K := \mathcal{O} \cap K$.

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Example 9

Consider the $\mathbb{Q}$-vector space $R = \mathbb{Q}[x]$ and quotient space $K := R/I$ where I is the subspace spanned by vector $\{(x^2 - 5) \cdot f : f \in R\}$. In other word,

$$I = \{f \in R : (x^2 - 5) \mid f\}$$

Then K is a number field, and it can be written as $\mathbb{Q}(\sqrt{5})$, i.e. K is isomorphic to the vector space spanned by two elements, 1 and $\sqrt{5}$ by the isomorphism $1 \mapsto 1$ and $x \mapsto \sqrt{5}$.

Ideals

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Example 12

Consider the number field $K = \mathbb{Q}(\sqrt{-5}) = \mathbb{Q}[x]/(x^2 + 5)$. One can show $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$ in this case, and the ideal 2 is not prime. To see this, note $2 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ while none of those two elements lies in the ideal

$$(2) := \{2a + 2b\sqrt{-5} : a, b \in \mathbb{Z}\}$$

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Remark

Note here product of ideals IJ is defined by

$$IJ := \{ab : a \in I, b \in J\}$$

S-Units

Definition 15

Let K be a number field, and S a finite set of prime ideals in $\mathcal{O}_K$. Then $\alpha \in \mathcal{O}_K$ is a ***S-unit*** if the principal ideal (α) is a product of prime ideals in S.

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Let K be a number field, m positive integer, S a finite set of primes in $\mathcal{O}_K$. Then for every $\epsilon > 0$ there exists constant C, depending on m, K, S and ϵ, such that, for any S-units $x_1, ..., x_m$ with no proper subsum equal 0, we have

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Remark

The proof of the above theorem uses a very vast generalization of what's called Roth's theorem, which is a field medal result. For a pointer, Roth's theorem admits a vast generalization called Schmidt's subspace theorem, and this theorem uses the p-adic version of subspace theorem.

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Theorem 17

Let $\mathbf{u}$ *be non-degenerate and simple. Then, there exists a function* $F : \mathbb{T}^s \to \mathbb{R}$*, depending on* $\mathbf{u}$*, so that* $\mathbf{u}$ *is ultimately positive iff* $F(\mathbf{z}) \geq 0$ *for all* $\mathbf{z} \in T(\mathbf{a}) \subseteq \mathbb{T}^s$*.*

Result

Definition 16

A LRS is ***non-degenerate*** if it does not have two distinct characteristic roots whose quotient is a root of unity.

Theorem 17

Let $\mathbf{u}$ *be non-degenerate and simple. Then, there exists a function* $F : \mathbb{T}^s \to \mathbb{R}$*, depending on* $\mathbf{u}$*, so that* $\mathbf{u}$ *is ultimately positive iff* $F(\mathbf{z}) \geq 0$ *for all* $\mathbf{z} \in T(\mathbf{a}) \subseteq \mathbb{T}^s$*.*

Here $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{T}^s$ is s copy of $\mathbb{T}$

Definitions

Suppose $\mathbf{u}$ is simple and non-degenerate, with dominant roots

$$\rho, \gamma_1, \ldots, \gamma_s, \overline{\gamma_1}, \ldots, \overline{\gamma_s}$$

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Then,

$$u_n = b\rho^n + \sum_{i=1}^{s} c_i\gamma_i^n + \overline{c_i\gamma_i}^n + r(n)$$

where $r(n)$ is relatively small.

Definitions

Now set $\lambda_i = \gamma_i/\rho$, then

$$u_n = \rho^n F(\lambda_1^n, ..., \lambda_s^n) + r(n)$$

where

$$F(z_1, ..., z_s) = b + c_1 z_1 + ... + c_s z_s + \overline{c_1 z_1} + ... + \overline{c_s z_s}$$

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Definition

Let $\mathbf{a} = (\lambda_1, ..., \lambda_s)$ as above, we define

$$L(\mathbf{a}) := \{(v_1, ..., v_s) \in \mathbb{Z}^s : a_1^{v_1} ... a_s^{v_s} = 1\} \subseteq (\mathbb{Z}^s, +)$$

$$T(\mathbf{a}) := \{(\mu_1, ..., \mu_s) \in \mathbb{T}^s : \mu_1^{v_1} ... \mu_s^{v_s} = 1 \text{ for all } \mathbf{v} \in L(\mathbf{a})\}$$

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This completes our main criterion for ultimate positivity when $\mathbf{u}$ is simple and non-degenerate

Sketch Proof

Recall our main criterion:

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Theorem 18

Let $\mathbf{u}$ *be non-degenerate and simple. Then, there exists a function* $F : \mathbb{T}^s \to \mathbb{R}$*, depending on* $\mathbf{u}$*, so that* $\mathbf{u}$ *is ultimately positive iff* $F(\mathbf{z}) \geq 0$ *for all* $\mathbf{z} \in T(\mathbf{a}) \subseteq \mathbb{T}^s$*.*

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Now we proceed on give a sketch proof

Sketch Proof

Let $\rho, \gamma_1, ..., \gamma_s, \overline{\gamma_1}, ..., \overline{\gamma_s}$ be the dominant roots of $f_{\mathbf{u}}$. Then we get

$$u_n = b\rho^n + \sum c_i\gamma_i^n + \overline{c_i\gamma_i^n} + r(n)$$

where $r(n) = o(\rho^{n(1-\epsilon)})$ for some $\epsilon > 0$

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Let $K/\mathbb{Q}$ be the number field generated by all the characteristic roots of $\mathbf{u}$, and S the set of prime ideal divisors of the dominant characteristic roots.

Sketch Proof

By construction

$$b\rho^n + \sum c_i\gamma_i^n + \overline{c_i\gamma_i^n}$$

is a sum of S-units. Thus apply the theorem on S-units:

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Theorem 19

Let K be a number field, m positive integer, S a finite set of primes in $\mathcal{O}_K$. Then for every $\epsilon > 0$ there exists constant C, depending on m, K, S and ϵ, such that, for any S-units $x_1, ..., x_m$ with no proper subsum equal 0, we have

$$|x_1 + ... + x_m| \geq CXY^{-\epsilon}$$

where X, Y are computable constants only depending on K and x_i.

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where X, Y are computable constants only depending on K and x_i.

Here $X = C_1\rho^n$, $Y = C_2\rho^n$ for some constant $C_1, C_2 > 0$. Thus for all $\epsilon > 0$ there exists constant C so

$$\left|b\rho^n + \sum c_i\gamma_i^n + \overline{c_i\gamma_i^n}\right| \geq C\rho^{n(1-\epsilon)}$$

for all but finitely many n

Sketch Proof

Now just pick the ϵ so $r(n) = o(\rho^{n(1-\epsilon)})$, we conclude the sum of dominant roots indeed dominant the whole sum for all but finitely many n.

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Thus u_n ultimately positive iff

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But then $b\rho^n + \sum c_i\gamma_i^n + \overline{c_i\gamma_i^n}$ can be re-write as $\rho^n F(\lambda_1^n, ..., \lambda_s^n)$, where the positivity of F can be checked on the torus, which is a computable task.

General Case

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1. Find subsequences $\mathbf{u}_i$ for $1 \leq i \leq M$ such that the union of $\mathbf{u}_i$ is equal $\mathbf{u}$, where all $\mathbf{u}_i$ are non-degenerate
2. Verify those $\mathbf{u}_i$ individually using the criterion above

Example 1

Suppose

$$u_n = \frac{3}{2}u_{n-1} + \frac{3}{2}u_{n-2} - u_{n-3}$$

Then

$$f_{\mathbf{u}} = (x-2)(x+1)(x-\frac{1}{2})$$

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Hence

$$\begin{aligned} u_n = 2^n &\left(\frac{8}{9}\,u_2 + \frac{4}{9}\,u_1 - \frac{4}{9}\,u_0\right) \\ &+ (-1)^n\left(\frac{2}{9}\,u_2 - \frac{5}{9}\,u_1 + \frac{2}{9}\,u_0\right) \\ &+ \frac{-\frac{1}{9}\,u_2 + \frac{1}{9}\,u_1 + \frac{2}{9}\,u_0}{2^n} \end{aligned}$$

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Conclusion

$u_n = \frac{3}{2}u_{n-1} + \frac{3}{2}u_{n-2} - u_{n-3}$ is ultimately positive if and only if (u_2, u_1, u_0) lies in a half-space defined by $(8/9, 4/9, -4/9)$.

Example 2

Consider

$$u_n = (2\sqrt{5} + 3)u_{n-1} - 3(2\sqrt{5} + 3)u_{n-2} - 27u_{n-3}$$

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$$u_n = (2\sqrt{5} + 3)u_{n-1} - 3(2\sqrt{5} + 3)u_{n-2} - 27u_{n-3}$$

Then

$$\begin{aligned} f_{\mathbf{u}} &= x^3 - \left(2\sqrt{5} + 3\right)x^2 + 3\left(2\sqrt{5} + 3\right)x - 27 \\ &= (x-3)(x-\gamma_1)(x-\gamma_2) \end{aligned}$$

where γ_1, γ_2 are conjugate to each other.

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1. the numerical instability is mostly coming from the error due to exponents
2. to compute $F(\mathbf{z})$ we do not need the exponents!

Example 2

After some computation, we obtain

$$\begin{aligned} F(z) \approx\ & 18u_0 - 9u_1 + 2u_2 \\ & + (-(9-3i)u_0 + (4-4i)u_1 - (0.5-1i)u_2)z \\ & + (-(9+3i)u_0 + (4+4i)u_1 - (0.5+1i)u_2)\overline{z} \end{aligned}$$

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So

$$T(\lambda_1) = \{z \in \mathbb{C} : |z| = 1, z^0 = 1\} = \mathbb{T}$$

Example 2

Therefore, to see if **u** is ultimately positive or not, we just need to find the values of u_0, u_1, u_2 such that

$$\min(F(\cos(t) + i\sin(t)) : 0 \leq t \leq 2\pi) \geq 0$$

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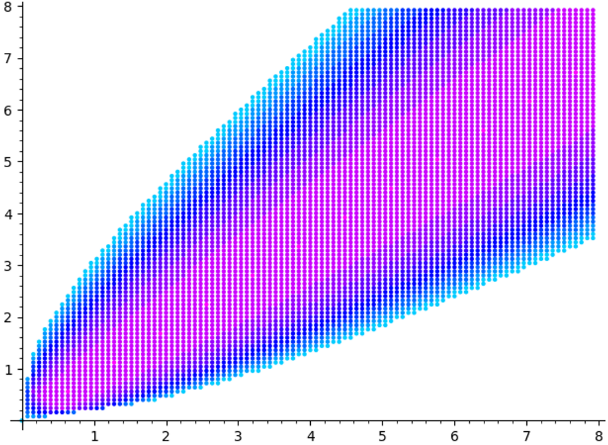
$$\min(F(\cos(t) + i\sin(t)) : 0 \leq t \leq 2\pi) \geq 0$$

In other word, we are minimizing

$$\begin{aligned} F(t) &= (9u_1 - 1u_2 - 18u_0)\cos(t) \\ &\quad + (8u_1 - 2u_2 - 7u_0)\sin(t) \\ &\quad - 9u_1 + 2u_2 + 18u_0 \\ &= A\cos(t) + B\sin(t) + C \end{aligned}$$

Example 2

We just give some plots:



Here (x, y)-axes are (u_1, u_2).

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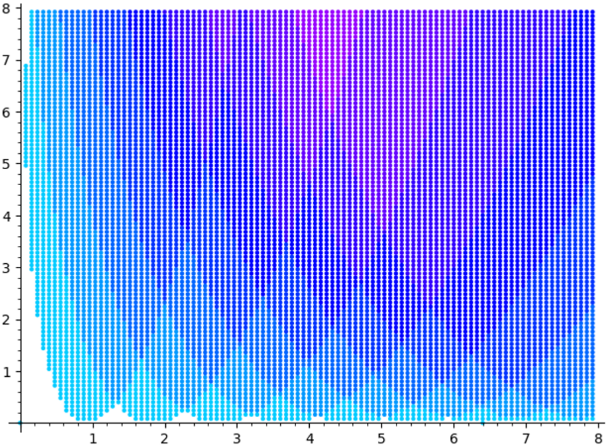

Here (x, y)-axes are (u_0, u_2).

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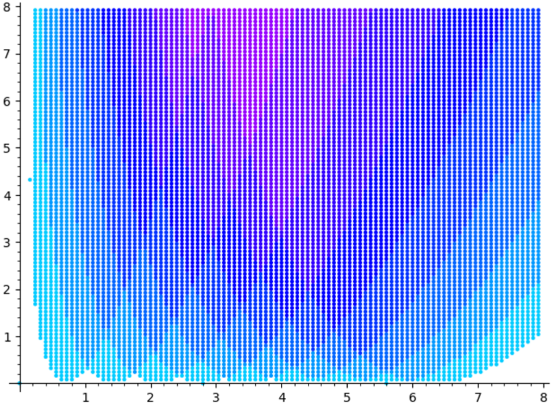

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