1 Reviews

Example 1.1

Let $V = \mathbb{R}^4$ and $W = \text{span}(e_1 + e_2 + e_3 + e_4)$. Find a basis of V/W.

Proof. Just pick e_1, e_2, e_3 and we claim this is a basis. Well, note $\dim W = 1$, $\dim V = 4$, so $\dim(V/W) = 3$. Thus we just need to show $e_1 + W, e_2 + W$ and $e_3 + W$ spans, and we have a theorem which says if $\dim V = n$ then $\{v_1, \ldots, v_n\}$ spans iff $\{v_1, \ldots, v_n\}$ linearly independent. But then this is obvious, as $\{e_1, e_2, e_3, e_4\}$ spans V/W but $e_4 + W = -e_1 - e_2 - e_3 + W$. Thus we are done.

Example 1.2

Let A be the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Show $A^3 = 0$.

Proof. Compute.

Example 1.3

Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

Compute A^m for $m \geq 1$.

Proof. Observe $A = 2I + 3B$ where $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Observe $B^2 = 0$ as one can compute this. But then we also have $(2I)(3B) = (3B)(2I)$, we can use binomial theorem to conclude

$$(2I + 3B)^m = \sum_{i=0}^{m} \binom{m}{i} (2I)^{m-i} (3B)^i$$

where only $i = 0, 1$ would give $(3B)^i \neq 0$. In other word,

$$(2I + 3B)^m = 2^m I + \binom{m}{1} 2^{m-1} I(3B) = \begin{bmatrix} 2^m & (3m) \cdot 2^{m-1} \\ 0 & 2^m \end{bmatrix}$$

Example 1.4

A matrix A is idempotent if $A^2 = A$. Show n by n matrix A is idempotent if and only if $\text{rank}(A) + \text{rank}(I - A) = n$.

Proof. First note that we can do elementary row/column opeartions on block matrices as well. In particular, consider the block matrix

$$\begin{bmatrix} A & 0 \\ 0 & I - A \end{bmatrix}$$

and we add the first row $[A, 0]$ to the second row, and we get

$$\begin{bmatrix} A & 0 \\ A & I - A \end{bmatrix}$$

Now add the first column to the second column, we get

$$\begin{bmatrix} A & A \\ A & I \end{bmatrix}$$

Next, multiply second row by $-A$ and add to the first row, we get

$$\begin{bmatrix} A - A^2 & 0 \\ A & I \end{bmatrix}$$

Multiply second column by $-A$ and add to the first column, we get

$$\begin{bmatrix} A - A^2 & 0 \\ 0 & I \end{bmatrix}$$

This shows

$$\text{rank}\begin{bmatrix} A & 0 \\ 0 & I - A \end{bmatrix} = \text{rank}\begin{bmatrix} A - A^2 & 0 \\ 0 & I \end{bmatrix}$$

Convenience yourself ranks are additive on block matrices, i.e. we get

$$\text{rank}(A) + \text{rank}(I - A) = \text{rank}(A - A^2) + \text{rank}(I_n) = n$$

We are done.

Example 1.5

Let $T : V \to V$ be linear transformation with $\dim V = n$. Prove that

$$\text{rank}(T^n) = \text{rank}(T^{n+k})$$

for all $k \geq 1$.

Proof. If T is invertible then T^m is invertible for all $m \geq 1$ and in particular $\text{rank}(T^m) = n$ for all $m \geq 1$.

Thus now assume T is not invertible. In this case, $\text{rank}(T) < n$. But observe that

$$\text{rank}(T) \geq \text{rank}(T^2) \geq \text{rank}(T^3) \geq \ldots \geq \text{rank}(T^n) \geq \text{rank}(T^{n+1})$$

This is $n+1$ integers less than n, and thus by Pigeonhole we get some $m < n+1$ such that the $\geq$ is in fact $=$, i.e.

$$\text{rank}(T^m) = \text{rank}(T^{m+1}) \Rightarrow \text{im}(T^m) = \text{im}(T^{m+1})$$

But this implies $\text{rank}(T^m) = \text{rank}(T^{m+k})$ for all $k \geq 1$. Indeed, observe

$$\begin{aligned}\text{im}(T^{m+1}) &= \{TT^m x : x \in V\} \\ &= \{Tx : x \in \text{im}(T^m)\} \\ &= \{Tx : x \in \text{Im}(T^{m+1})\} = \text{im}(T^{m+2})\end{aligned}$$

and now use induction we are done.

2 Enrichment:Permutation 1

Definition 2.1

A permutation is a automorphism of sets between $[n]$, i.e. $\sigma \in S_n$ if $\sigma : [n] \to [n]$ is a bijection

Example 2.2

Bijection of sets is just a bijection of sets... For example, $\sigma : [4] \to [4]$ defined by $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 3$ and $\sigma(4) = 1$ is a permutation.

We can also compose permutations, it is just composition of functions...

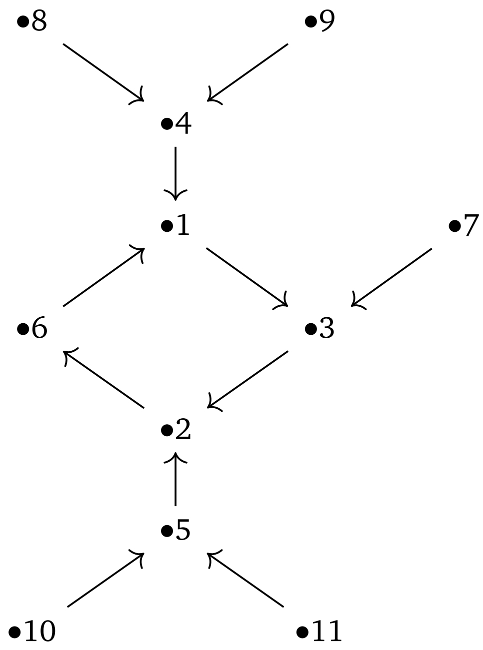
Example 2.3

Let σ be in the last example, and $\tau(1) = 2$, $\tau(2) = 3$, $\tau(3) = 4$ and $\tau(4) = 1$. Then we see $\sigma\tau(1) = 3$, $\sigma\tau(2) = 2$, $\sigma\tau(3) = 1$ and $\sigma\tau(4) = 4$.

Construction 2.4

Let σ be any endofunction of $[n]$, i.e. any function $\sigma : [n] \to [n]$, we can associate a directed graph to σ by define vertices as $[n]$ and edges as $(x, \sigma(x))$. In particular, those two things are the "same", i.e. one endofunction defines a unique directed graph with out-number 1 and every directed graph with out-number 1 defines an endofunction.

Here is an example:



This directed graph clearly defines a endofunction. In fact, this endofunction is 2-to-1, i.e. $\sigma^{-1}(x)$ is always exactly two elements.

Since permutations are endofunctions, we get directed graphs out of it.

Let's consider some basic properties of this kind of directed graph.

Definition 2.5

Let G be a directed graph, a ***(directed) cycle*** is a path which only first and last vertices are equal.

Proposition 2.6

Let σ be a permutation, G be its associated directed graph. For $x \in [n]$ define $O(x) = \{\sigma^d(x) : d \geq 0\}$. Then $\{(x, \sigma^d(x)) : d \geq 0\}$ is a directed cycle.

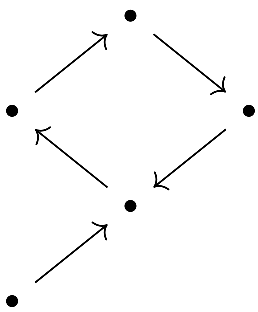
Proof. First note $O(x)$ must be a finite set since $\sigma : [n] \to [n]$. In particular, this means $\{x, \sigma(x), \sigma^2(x), \ldots\}$ must start to repeat at some point, Say $\sigma^{m+1}(x) \in \{\sigma^i(x) : 0 \leq i \leq m\}$ where $m+1$ is the minimal such element, then $\sigma^{m+1}(x) = \sigma^k(x)$ with $k < m+1$. Now σ is a permutation, thus the inverse exists, and we get $\sigma^{m+1-k}(x) = x$. This forces k to be zero as $m+1$ has to be minimal, and hence we indeed get a cycle. It is obvious this is a directed cycle.

Proposition 2.7

Let $x, y \in [n]$, then $O(x) \cap O(y) = \emptyset$ or $O(x) = O(y)$.

Proof. Well, suppose $z \in O(x) \cap O(y)$, this means $\sigma^k(x) = \sigma^m(y) = z$. If $k = m$ then $x = y = \sigma^{-k}(z)$ and so $O(x) = O(y)$ as desired. Now WLOG assume $k < m$, and so we get $x = \sigma^{m-k}(y) = \sigma^{-k}(z)$. But then $x \in O(y)$ and so $O(x) \subseteq O(y)$ where both $O(x)$ and $O(y)$ are cycles. But then there is only one possible sub-cycle in a cycle, which is the cycle itself, i.e. $O(x)$ must equal $O(y)$.

Next, note it is impossible for permutations to have the following situation



as σ is bijection means only one arrow goes in and out the same node.

This obversation plus the propositions shows the following theorem:

Theorem 2.8

The directed graph of σ is a disjoint union of directed cycles.

Well, why do we care? Observe directed cycles correspond to a permutation that moves one set $S \subseteq [n]$ around, and fix the rest. For example, $\tau(1) = 2$, $\tau(2) = 3$, $\tau(3) = 4$ and $\tau(4) = 1$ is a cycle. In general, cycles are defined by $(n_1, n_2, ..., n_k)$ where this notation means a permutation that sends n_1 to n_2, n_2 to n_3 and at the end n_k to n_1, while fix all other elements.

Two cycles $\sigma = (n_1, ..., n_k)$ and $\tau = (m_1, ..., m_p)$ are disjoint if the set $\{n_1, ..., n_k\}$ and $\{m_1, ..., m_p\}$ are disjoint.

What we just proved is the following:

Theorem 2.9

Every permutation admits a disjoint cycle decomposition, and it is unique up to reoredering. In other word, every permutation is a set of cycles.

Now let's just do some examples.

Example 2.10

Consider $\sigma : [4] \to [4]$ defined by $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 3$ and $\sigma(4) = 1$. THen this decompose as $(124)(3)$.

Next, if $\tau = 48635127$ where we used one-line notation, i.e. $4 = \tau(1)$, $8 = \tau(2)$ and so on. Then $\tau = (1436)(287)(5)$.

If we still have time:

Example 2.11

Let σ be a $2n+1$ permutation such that $\sigma(1) > \sigma(2) < \sigma(3) > \dots\pi(2n) < \pi_{2n+1}$. Let t_n be the number of such permutations (in particular if n is even then $t_n = 0$). Show that

$$T(x) := \sum_{n\geq 0} \frac{t_n}{n!} x^n = \tan(x)$$

Proof. Only a sketch proof.

Step 1: show that $t_{2k+1} = \sum_{1\leq j\leq 2k, j \text{ odd}} \binom{2j}{j} a_j a_{2k-j}$. This is easy: we just delete the 1 in our one-line notation sequence, and turn those two substrings into two new permutations. This is not 1-to-1, and we must add the factor $\binom{2j}{j}$, which concludes the proof.

Step 2: this recurrence implies we get

$$T'(x) = T(x)^2 + 1$$

and solve for it we get $\tan(x)$.