1 Reviews

Example 1.1

Let $V = \mathbb{R}^4$ and $W = \text{span}(e_1 + e_2 + e_3 + e_4)$. Find a basis of V/W.

Proof. Just pick e_1, e_2, e_3 and we claim this is a basis. Well, note $\dim W = 1$, $\dim V = 4$, so $\dim(V/W) = 3$. Thus we just need to show $e_1 + W, e_2 + W$ and $e_3 + W$ spans, and we have a theorem which says if $\dim V = n$ then $\{v_1, ..., v_n\}$ spans iff $\{v_1, ..., v_n\}$ linearly independent. But then this is obvious, as $\{e_1, e_2, e_3, e_4\}$ spans V/W but $e_4 + W = -e_1 - e_2 - e_3 + W$. Thus we are done.

Example 1.2

Let A be the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Show $A^3 = 0$.

Proof. Compute.

Example 1.3

Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

Compute A^m for $m \geq 1$.

Proof. Observe $A = 2I + 3B$ where $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Observe $B^2 = 0$ as one can compute this. But then we also have $(2I)(3B) = (3B)(2I)$, we can use binomial theorem to conclude

$$(2I + 3B)^m = \sum_{i=0}^{m} \binom{m}{i} (2I)^{m-i} (3B)^i$$

where only $i = 0, 1$ would give $(3B)^i \neq 0$. In other word,

$$(2I + 3B)^m = 2^m I + \binom{m}{1} 2^{m-1} I(3B) = \begin{bmatrix} 2^m & (3m) \cdot 2^{m-1} \\ 0 & 2^m \end{bmatrix}$$

Example 1.4

A matrix A is idempotent if $A^2 = A$. Show n by n matrix A is idempotent if and only if $\text{rank}(A) + \text{rank}(I - A) = n$.

Proof. First note that we can do elementary row/column opeartions on block matrices as well. In particular, consider the block matrix

$$\begin{bmatrix} A & 0 \\ 0 & I - A \end{bmatrix}$$

and we add the first row $[A, 0]$ to the second row, and we get

$$\begin{bmatrix} A & 0 \\ A & I - A \end{bmatrix}$$

Now add the first column to the second column, we get

$$\begin{bmatrix} A & A \\ A & I \end{bmatrix}$$

Next, multiply second row by $-A$ and add to the first row, we get

$$\begin{bmatrix} A - A^2 & 0 \\ A & I \end{bmatrix}$$

Multiply second column by $-A$ and add to the first column, we get

$$\begin{bmatrix} A - A^2 & 0 \\ 0 & I \end{bmatrix}$$

This shows

$$\text{rank}\begin{bmatrix} A & 0 \\ 0 & I - A \end{bmatrix} = \text{rank}\begin{bmatrix} A - A^2 & 0 \\ 0 & I \end{bmatrix}$$

Convenience yourself ranks are additive on block matrices, i.e. we get

$$\text{rank}(A) + \text{rank}(I - A) = \text{rank}(A - A^2) + \text{rank}(I_n) = n$$

We are done.

Example 1.5

Let $T : V \to V$ be linear transformation with $\dim V = n$. Prove that

$$\text{rank}(T^n) = \text{rank}(T^{n+k})$$

for all $k \geq 1$.

Proof. If T is invertible then T^m is invertible for all $m \geq 1$ and in particular $\text{rank}(T^m) = n$ for all $m \geq 1$.

Thus now assume T is not invertible. In this case, $\text{rank}(T) < n$. But observe that

$$\text{rank}(T) \geq \text{rank}(T^2) \geq \text{rank}(T^3) \geq \ldots \geq \text{rank}(T^n) \geq \text{rank}(T^{n+1})$$

This is $n+1$ integers less than n, and thus by Pigeonhole we get some $m < n+1$ such that the $\geq$ is in fact $=$, i.e.

$$\text{rank}(T^m) = \text{rank}(T^{m+1}) \Rightarrow \text{im}(T^m) = \text{im}(T^{m+1})$$

But this implies $\text{rank}(T^m) = \text{rank}(T^{m+k})$ for all $k \geq 1$. Indeed, observe

$$\begin{aligned} \text{im}(T^{m+1}) &= \{TT^m x : x \in V\} \\ &= \{Tx : x \in \text{im}(T^m)\} \\ &= \{Tx : x \in \text{Im}(T^{m+1})\} = \text{im}(T^{m+2}) \end{aligned}$$

and now use induction we are done.

2 Permutation Matrices

Today our goal is the following: permutations too hard, we want linear algebra

Let σ be a n-permutation, then we can define

$$P_\sigma := \begin{bmatrix} e_{\sigma^{-1}(1)} \\ e_{\sigma^{-1}(2)} \\ \vdots \\ e_{\sigma^{-1}(n)} \end{bmatrix}$$

here e_i are the row standard basis of $\mathbb{F}^n$. For example, if $\sigma = (1,2,3)$ then we see

$$P_\sigma = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Next, let $\sigma = (124)$, then

$$P_\sigma = \begin{bmatrix} e_4 \\ e_1 \\ e_3 \\ e_2 \end{bmatrix} = [3]$$

In general, we have:

1. P_σ has one 1 in each row and in each column, all other entries are 0

2. The ith row of P_σ is $e_{\sigma^{-1}(i)}$ and the ith column is $e_{\sigma(1)}$
3. For any matrix A, $P_\sigma A$ moves the ith row of A to the $\sigma(i)$th row.
4. For any matrix B, BP_σ moves the ith column of B to the $\sigma(i)$th column

Next, we are just going to do some more detailed study of S_n.

Definition 2.1

A subgroup of S_n is a subset $G \subseteq S_n$ such that:

1. $\mathrm{Id} \in G$
2. $\sigma \in G$ then $\sigma^{-1} \in G$
3. $\sigma, \tau \in G$ then $\sigma \circ \tau \in G$

Example 2.2

Let $G = \{(123), (123)^2, (123)^3\}$, then one can verify it forms a subgroup. Indeed, note $(123)^2 = (132)$ and $(123)^3 = \mathrm{Id}$. Thus $(123)^n = (123)^m$ where $0 \leq m < 3$ and $n \equiv m \pmod 3$.

Definition 2.3

Let X be a finite set and G a subgroup of S_n. We say X ***admits a G-action*** if there exists a set function $\alpha : G \times X \to X$ such that:

1. $\alpha(\mathrm{Id}, x) = x$ for all $x \in X$
2. $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$

If this is the case, we say X is a (left) G-set.

From now on we write $g \cdot x$ to mean $\alpha(g, x)$.

Definition 2.4

Let X be a G-set, for $x \in X$, we define the ***stabilizer*** $\mathrm{Stab}(x) = \{g \in G : gx = x\} \subseteq G$ and ***orbit*** $\mathrm{Orb}(x) := \{gx : g \in G\} \subseteq X$.

Proposition 2.5

Let X be G-set, then $\mathrm{Orb}(x) = \mathrm{Orb}(y)$ or $\mathrm{Orb}(x) \cap \mathrm{Orb}(y) = \emptyset$.

Proof. We will show $\mathrm{Orb}(x) \cap \mathrm{Orb}(y) \neq \emptyset$ then $\mathrm{Orb}(x) = \mathrm{Orb}(y)$. Take $z \in \mathrm{Orb}(x) \cap \mathrm{Orb}(y)$, then $z = g_1 x = g_2 y$ and so $x = g_1^{-1} g_2 y$ which shows $\mathrm{Orb}(x) \subseteq \mathrm{Orb}(y)$. But then $y = g_2^{-1} g_1 x$, so $\mathrm{Orb}(y) \subseteq \mathrm{Orb}(x)$.

Theorem 2.6

Let X be a G-set. Then

$$|\{\mathrm{Orb}(x) : x \in X\}| = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|$$

where $\mathrm{Fix}(g) := \{x \in X : gx = x\}$.

Proof. Basic counting argument.

Example 2.7

Suppose we have n choices of colours to paint the four corners of a floor tile. How many different floor tiles can we make?

Proof. Let us label the corner of our tile by $1,2,3,4$ going clockwise. Then, two paintings are the same, iff we can rotate one colour configuration to get another. In other word, consider the subgroup $R := \{(1234),(1234)^2,(1234)^3,\mathrm{Id}\}$. This acts on the set $\{1,2,3,4\}$ just like the rotations. In other word, we get an R-action on the set of all possible colour configurations X. The question is exactly asking the number of orbits of X under action of R. But then by the above theorem, we see this is equal to

$$\frac{1}{|R|}\left(\mathrm{Fix}(\mathrm{Id}) + \mathrm{Fix}((1234)) + \mathrm{Fix}((1234)^2)) + \mathrm{Fix}((1234)^3)\right)$$

But $\mathrm{Fix}(\mathrm{Id}) = n^4$ because we can freely choose any colour on any of the four corners.

Next, $\mathrm{Fix}((1234)) = n$ because the colour of 1 must equal the colour of 2, and the colour of 2 must equal the colour of 3 and so on, i.e. we only get to choose one colour and we are done.

Then, $\mathrm{Fix}((1234)^2) = n^2$ and $\mathrm{Fix}((1234)^3) = n$.

Hence, we conclude we have

$$\frac{n^4 + n^2 + 2n}{4}$$

many possibilities.