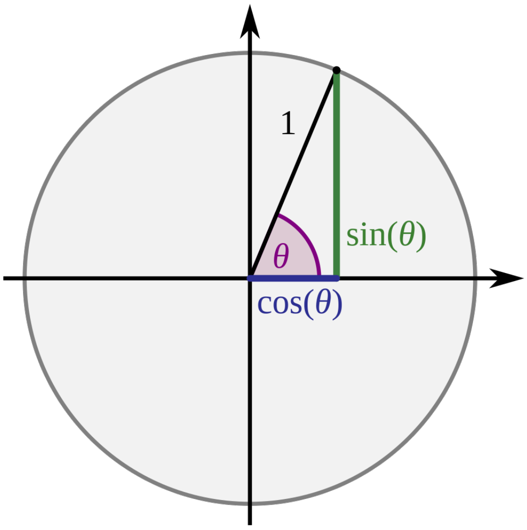
1 Examples

Example 1.1

Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation obtained by rotate vector v θ radians counterclockwise, where $0 \leq \theta \leq 2\pi$. Then:

1. find $[R]_{\mathcal{E}}$.
2. show that if $\theta \in (0, 2\pi)$ then there exists no non-zero vector v such that $Rv = \lambda v$ for some $\lambda \in \mathbb{R}$.

For the first one, note



In other word, we see $(1,0) \mapsto v$ such that v has angle θ, i.e. by the definition of cos and sin, we get

$$v = (\cos(\theta), \sin(\theta))$$

On the other hand, we see $(0,1) \mapsto u$ where u has angle $\pi/2 + \theta$, i.e.

$$u = (\cos(\pi/2 + \theta), \sin(\pi/2 + \theta))$$

Basic calculus now tells us

$$u = (-\sin\theta, \cos\theta)$$

Hence, we concluded

$$[R]_{\mathcal{E}} = \begin{bmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

To show (2), observe $\exists x \neq 0, Ax = \lambda x$ if and only if $(A - \lambda I)x = 0$ if and only if $\ker(A - \lambda I) \neq 0$ if and only if $A - \lambda I$ is not full rank. Hence it suffices to show $[R]_{\mathcal{E}} - \lambda I$ is full rank for any choice of λ.

Well, first we show

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$. To do this, let's find the formal inverse of the matrix A. That is, we are looking for

$$B = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

so

$$AB = tI = BA$$

where $t \neq 0$. Thus we get

$$\begin{bmatrix} ax_1 + bx_2 & ax_3 + bx_4 \\ cx_1 + dx_3 & cx_3 + dx_4 \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} = \begin{bmatrix} ax_1 + cx_2 & bx_1 + dx_2 \\ ax_3 + cx_4 & bx_3 + dx_4 \end{bmatrix}$$

Compare entries we are just solving for linear equations in terms of x_1, x_2, x_3, x_4, and at the end we conclude

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus, if the inverse of A exists, it must equal B, and in particular forces $ad - bc$ to be non-zero. On the other hand, if $ad - bc$ is non-zero then B exists and hence A has inverse.

Hence, to show

$$[R]_{\mathcal{E}} - \lambda I = \begin{bmatrix} \cos(\theta) - \lambda & -\sin\theta \\ \sin(\theta) & \cos(\theta) - \lambda \end{bmatrix}$$

is of full rank, it suffices to consider

$$\begin{aligned} (\cos(\theta) - \lambda)^2 + \sin^2(\theta) &= \cos^2(\theta) - 2\cos(\theta)\lambda + \lambda^2 + \sin^2(\theta) \\ &= \lambda^2 - 2\cos(\theta)\lambda + 1 \end{aligned}$$

This has solution iff

$$4\cos^2(\theta) - 4 = 0 \Leftrightarrow \cos(\theta) = 0$$

but we assumed $\theta \in (0, 2\pi)$.

Fun Fact 1.2

A proper rigid transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a set map such that

$$||x - y|| = ||T(x) - T(y)||$$

for all $x, y \in \mathbb{R}^2$, and $T(e_1)$ is always at the right of $T(e_2)$ (this is called the handedness/orientation). Then, we can prove all such rigid transformation are of the form $f(x) = Rx + v$ where R is a rotation matrix as above, and v is a fixed vector. In other word, all proper rigid transformation are defined by rotation plus translation.

Quiz Q1 1.3

In assignment 5, is the linear transformation in Problem 2 a proper rigid transformation?

Recall the map R is defined by reflection about the line $y = 2x$.

Example 1.4

Let f, g be polynomials, with $\deg(g) = m$ and $\deg(f) = n$, in $\mathbb{C}[x]$. Prove that $f(x)$ and $g(x)$ has common root implies $\text{Res}(f, g)$ is zero.

The definition of $\text{Res}(f, g)$ is

$$\begin{vmatrix} a_n & a_{n-1} & \dots & a_0 & & & & \\ & a_n & a_{n-1} & \dots & a_0 & & & \\ & & \dots & \dots & \dots & \dots & & \\ & & & & a_n & a_{n-1} & \dots & a_0 \\ b_m & b_{m-1} & \dots & b_0 & & & & \\ & b_m & b_{m-1} & \dots & b_0 & & & \\ & & \dots & \dots & \dots & \dots & & \\ & & & & b_m & b_{m-1} & \dots & b_0 \end{vmatrix}$$

where we get m rows of $(a_n, \dots, a_0)$ and n rows of $(b_m, \dots, b_0)$. Here $f = a_n x^n + \dots$ and $g = b_m x^m + \dots$.

For example, if $f(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$ and $g(x) = 2x^3 + x^2 + 3x + 4$, then we have

$$R(f, g) = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 2 & 1 & 3 & 4 \end{vmatrix}$$

Before we start, note if f or g are the zero polynomial then this is trivial.

First, let's suppose f, g has a common root, thus $f(x) = f_1(x)d(x)$ and $g(x) = g_1(x)d(x)$ where $\deg(d(x)) \geq 1$. Thus we see

$$g_1(x)f(x) = f_1(x)g(x) = d(x)$$

Now set $g_1(x) = d_{m-1}x^{m-1} + \dots + d_0$ and $f_1(x) = c_{n-1}x^{n-1} + \dots + c_0$, we see

$$g_1(x)f(x) = f_1(x)g(x) \Rightarrow a_n d_{m-1} = b_m c_{n-1}$$

by compare the leading coefficient. Similarly, by compare the second term, we get

$$a_{n-1}d_{m-1} + a_n d_{m-2} = b_{m-1}c_{n-1} + b_m c_{n-2}$$

Its not hard to see, we get a system of linear equations

$$\sum_{i+j=m+n-k} a_i d_j = \sum_{i+j=m+n-k} b_i c_j, \quad 1 \leq k \leq m+n-1$$

Since f, g are not zero, we see f_1, g_1 cannot be zero polynomials. Hence we immediately see $(d_{m-1}, \dots, d_0, -c_{n-1}, \dots, -c_0)$ is a solution to the matrix of $\text{Res}(f, g)$. In other word, it is not invertible, and thus $\text{Res}(f, g)$ is zero as desired.

Fun Fact 1.5

This condition is if and only if.

Example 1.6

Let $f_1, \dots, f_n$ be $(n-1)$ times differentiable real functions on interval $[a, b]$. Define the Wronskian by

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ \dots & \dots & \dots & \dots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix}$$

Prove $f_1, \dots, f_n$ are linearly independent over $C^{(n-1)}[a, b]$ as $\mathbb{R}$-vector space if there exists $x_0 \in [a, b]$ so $W(x_0) \neq 0$.

Suppose

$$\sum k_i f_i(x) = 0$$

for some $k_i \in \mathbb{R}$. Then take derivatives on both sides, we get

$$\sum k_i f_i'(x) = 0$$

Do this $n-1$ times, we get

$$\begin{cases} \sum k_i f_i(x) = 0 \\ \sum k_i f_i'(x) = 0 \\ \sum k_i f_i^{(2)}(x) = 0 \\ \vdots \\ \sum k_i f_i^{(n-1)}(x) = 0 \end{cases}$$

But then if we subsitute $x = x_0$ we see we get

$$\begin{bmatrix} f_1(x_0) & f_2(x_0) & \dots & f_n(x_0) \\ f_1'(x_0) & f_2'(x_0) & \dots & f_n'(x_0) \\ \dots & \dots & \dots & \dots \\ f_1^{(n-1)}(x_0) & f_2^{(n-1)}(x_0) & \dots & f_n^{(n-1)}(x_n) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = 0$$

In other word, $(k_1, \dots, k_n)$ is in the kernel of the matrix on the left. But then $W(x_0)$ is the determinant of that matrix, and by assumption it is not zero. Thus we see $(k_1, \dots, k_n)$ has to be the zero vector. This concludes $f_1, \dots, f_n$ are linearly independent.

Quiz Q2 1.7

In the $\mathbb{R}$-vector space of functions from $\mathbb{R}$ to $\mathbb{R}$, is the set $\{x^2, x \cdot |x|\}$ linearly independent?

Note they are linearly independent but its Wronskian is actually always zero.

Quiz Q3 1.8

Is the $\mathbb{R}$-vector space of functions from $\mathbb{R}$ to $\mathbb{R}$, is the set

$$\{\sin(x), \cos(x), e^{\cos(x)} - 3\sin(x)\}$$

linearly independent?

Its Wronskian is

$$\begin{vmatrix} \sin(x) & \cos(x) & e^{\cos(x)} - 3\sin(x) \\ \cos(x) & -\sin(x) & \cos(x)e^{\cos(x)} - 3\cos(x) \\ -\sin(x) & -\cos(x) & -\sin(x)e^{\cos(x)} + \cos^2(x)e^{\cos(x)} + 3\sin(x) \end{vmatrix}$$

Now just pick $x = 0$ and its determinant is non-zero, as we are computing

$$\begin{vmatrix} 0 & 1 & e \\ 1 & 0 & e-3 \\ 0 & -1 & e \end{vmatrix} = -2e$$

Example 1.9

Prove $\dim_{\mathbb{Q}}(\mathbb{R}) = \infty$

We begin with the following claim: $1, \sqrt[n]{3}, \sqrt[n]{3^2}..., \sqrt[n]{3^{n-1}}$ are linearly independent over $\mathbb{Q}$. Indeed, suppose otherwise, then we can find a_i not all zero so

$$a_0 + a_1\sqrt[n]{3} +,,, + a_{n-1}\sqrt[n]{3} = 0$$

Thus let $f(x) = \sum_{i=1}^{n-1} a_i x^i$ and we see $f(\sqrt[n]{3}) = 0$, i.e. $\sqrt[n]{3}$ is a root of f. On the other hand, we see $g(x) = x^n - 3$ also has $\sqrt[n]{3}$ as a real root. Hence, $f(x)$ and $g(x)$ as real polynomials we get f, g has a common factor $x - \sqrt[n]{3}$, i.e. f, g are not coprime. Now recall Eisenstein Criterion:

Theorem 1.10

Let $g(x)$ be polynomial with integer coefficients. Then $g(x) = \sum_{i=0}^{n} a_i x^n$ is irreducible if there exists prime p so:

1. *p divides a_i for each $0 \leq i < n$*
2. *p does not divide a_n*
3. *p^2 does not divide a_0*

By this, we know $g(x)$ is irreducible over $\mathbb{Q}$, thus we must have $g(x) \mid f(x)$ in $\mathbb{Q}[x]$ as $g(x)$ is irreducible and f, g has common factors. This is a contradiction to the fact $\deg(g) = n$.

Now, suppose for a contradiction $\mathbb{R}$ is finite dimensional $\mathbb{Q}$-vector space. Then say $\dim_{\mathbb{Q}}(\mathbb{R}) = n$, which forces any $n+1$ many vectors to be linearly dependent. But then take $1, \sqrt[n+1]{3}, ..., \sqrt[n+1]{3^n}$, which makes the claim a contradiction.

2 Enrichment:Determinant

Today we are going to give an alternative definition of determinant. To begin with, recall we defined permutations as bijections of $[n]$. Next, we learned that permutation matrices permutes the row or columns of matrix A, if we multiple P_σ from the left or right. Moreover, $P_\sigma P_\tau = P_{\sigma\tau}$ and $P_\sigma^{-1} = P_{\sigma^{-1}}$.

Lemma 2.1

For all $\sigma \in S_n$, $\det(P_\sigma) \in \{1, -1\}$.

Proof. Each P_σ is obtained by row swaps of Id. But then those elementary row operations change det by a factor of -1. Hence $\det P_\sigma \in \{-1, 1\}$ as desired.

Definition 2.2

For $\sigma \in S_n$, we define:

1. The ***sign*** of σ as $\operatorname{sgn}(\sigma) := \det(P_\sigma)$
2. The ***parity*** of σ is ***even*** if $\operatorname{sgn}(\sigma) = 1$, and ***odd*** otherwise.

How do we compute the sign of σ? Well, from the proof above, we see we just need to figure out the number of row swaps. Thus, if we let $\sigma = c_1 \ldots c_t$ where c_i are cycles, then each cycle clearly requires $\ell(c_i) - 1$ many row-swaps. Hence

$$\operatorname{sgn}(\sigma) = (-1)^{\sum(\ell(c_i)-1)}$$

Alternatively, this can also be computed using what's called inversion number $N(\sigma)$. A pair (i, j) is called an inversion of σ if $1 \le i < j \le n$ and $\sigma(i) > \sigma(j)$. Then $\operatorname{sgn}(\sigma) = (-1)^{N(\sigma)}$.

Example 2.3

Consider the permutation $\sigma = 48635127$. Then:

1. $\sigma = (1436)(287)(5)$ and hence $\operatorname{sgn}(\sigma) = (-1)^{3+2+0}$
2. You can try to list all the inversions. The answer is 17 and hence $\operatorname{sgn}(\sigma) = (-1)^{17}$

Now, use linearity of det on the first row, we get

$$\det(A) = \sum_{i=1}^{n} a_{1i} \begin{bmatrix} e_i \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

But then we can expand each of those new matrices again on the second row, and so on. At the end of the day, we get

$$\det(A) = \sum_{i_1,\ldots,i_n=1}^{n} a_{1i_1}a_{2i_2}\ldots a_{ni_n} \det \begin{bmatrix} e_{i_1} \\ \vdots \\ e_{i_n} \end{bmatrix}$$

but then if $i_1\ldots i_n$ is not a permutation, then we have matrix with two row equal, and hence det is zero. Therefore, we only need to consider permutations, i.e.

$$\det(A) = \sum_{\sigma\in S_n} a_{1\sigma(1)}\ldots a_{n\sigma(n)} \det \begin{bmatrix} e_{\sigma(1)} \\ \vdots \\ e_{\sigma(n)} \end{bmatrix} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma^{-1}) \prod_{i=1}^{n} a_{i\sigma(i)}$$

It is not hard to see this is the same as

$$\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)}\ldots a_{n\sigma(n)}$$

Thus we obtained the permutation definition of determinants.

3 Definition Of Determinant

I will include the definition of determinant here, just in case its not covered yet.

Definition 3.1

We define the determinant recursively $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ as follows:

1. If $A = (a)$ then $\det(A) = a$
2. If $A = (a_{ij})$ then

$$\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + \ldots = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \cdot \det(A_{i1})$$

   where A_{ij} is A delete ith row and jth column.

The following aresome basic properties of det:

Theorem 3.2

Determinant has the following properties:

1. $\det(A) = -\det(B)$ *if B is obtained by swap i and $i+1$th row of A*
2. $\det(A) = \det(A_1) + \det(A_2)$ *if* $A = (\ldots, u+v, \ldots)$ *and* $A_1 = (\ldots, u, \ldots)$ *and* $A_2 = (\ldots, v, \ldots)$.
3. $\det(AB) = \det(A)\det(B)$
4. $\det(A) \neq 0$ *iff A is invertible*