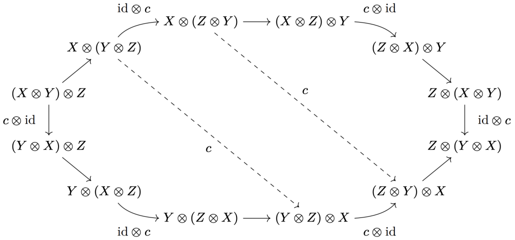
MATH 146 Section 2 Enrichment

Dongshu Dai

University Of Waterloo

January 19, 2023



Application Of Matrices

Let's find the general formula for Fibonacci sequence.

Application Of Matrices

Let's find the general formula for Fibonacci sequence.

Definition 1

Let $F_0 = 0$, $F_1 = 1$ and define $F_n = F_{n-1} + F_{n-2}$. We call the sequence $\{F_n\}_{n\geq 0}$ as the Fibonacci sequence.

Application Of Matrices

Let's find the general formula for Fibonacci sequence.

Definition 1

Let $F_0 = 0$, $F_1 = 1$ and define $F_n = F_{n-1} + F_{n-2}$. We call the sequence $\{F_n\}_{n\geq 0}$ as the Fibonacci sequence.

Question

Can we find a function $f : \mathbb{N} \to \mathbb{N}$ so that $F_n = f(n)$?

Observation

Observation

1. Since $F_n = F_{n-1} + F_{n-2}$ and $F_{n-1} = F_{n-1}$, we see F_n and F_{n-1} can be both written as linear combination of F_{n-1} and F_{n-2}.

Observation

1. Since $F_n = F_{n-1} + F_{n-2}$ and $F_{n-1} = F_{n-1}$, we see F_n and F_{n-1} can be both written as linear combination of F_{n-1} and F_{n-2}.
2. Thus we see for $n \geq 1$ we get

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$$

Observation

1. Since $F_n = F_{n-1} + F_{n-2}$ and $F_{n-1} = F_{n-1}$, we see F_n and F_{n-1} can be both written as linear combination of F_{n-1} and F_{n-2}.
2. Thus we see for $n \geq 1$ we get

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$$

3. Why stop here when we can repeat this? Clearly

$$\begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-2} \\ F_{n-3} \end{bmatrix}$$

Observation

1. Since $F_n = F_{n-1} + F_{n-2}$ and $F_{n-1} = F_{n-1}$, we see F_n and F_{n-1} can be both written as linear combination of F_{n-1} and F_{n-2}.
2. Thus we see for $n \geq 1$ we get

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$$

3. Why stop here when we can repeat this? Clearly

$$\begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-2} \\ F_{n-3} \end{bmatrix}$$

4. At the end of the day, we get

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

New Goal

For the sake of writing less, we denote

$$A := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

New Goal

For the sake of writing less, we denote

$$A := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Question

How to compute A^{n-1}?

Observation

Let's study this matrix a little bit closer.

Observation

Let's study this matrix a little bit closer.

1. Let's see how A acts on the standard basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Observation

Let's study this matrix a little bit closer.

1. Let's see how A acts on the standard basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. Clearly $Ae_1 = e_1 + e_2$ and $Ae_2 = e_1$

Observation

Let's study this matrix a little bit closer.

1. Let's see how A acts on the standard basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. Clearly $Ae_1 = e_1 + e_2$ and $Ae_2 = e_1$
3. In other word, under the action of A, both of the two standard vectors are rotated.

Observation

Let's study this matrix a little bit closer.

1. Let's see how A acts on the standard basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. Clearly $Ae_1 = e_1 + e_2$ and $Ae_2 = e_1$
3. In other word, under the action of A, both of the two standard vectors are rotated.
4. What if there are some vectors that are not rotated but only stretched?

Visualization

http://wosugi.sakura.ne.jp/app/linear-transform/

https://shad.io/MatVis/

Why That Matters?

Let $T : V \to V$ be a linear transformation, and if $Tv = \lambda v$, then we see

$$T^n v = \lambda^n v$$

Why That Matters?

Let $T : V \to V$ be a linear transformation, and if $Tv = \lambda v$, then we see

$$T^n v = \lambda^n v$$

Taking power is now no-brainer!

Why That Matters?

Let $T : V \to V$ be a linear transformation, and if $Tv = \lambda v$, then we see

$$T^n v = \lambda^n v$$

Taking power is now no-brainer!

1. If we can find a basis of $\mathbb{R}^2$ such that $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$

Why That Matters?

Let $T : V \to V$ be a linear transformation, and if $Tv = \lambda v$, then we see

$$T^n v = \lambda^n v$$

Taking power is now no-brainer!

1. If we can find a basis of $\mathbb{R}^2$ such that $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$
2. Then we see for any $v = a_1 v_1 + a_2 v_2$, we have

$$A^n v = A^n\left(\sum a_i v_i\right) = \sum a_i \lambda_i^n v_i$$

New Goal

Question

How do we find those magic vectors and magic numbers?

New Goal

Question

How do we find those magic vectors and magic numbers?

Answer

Well, determinant is not covered yet, so I can't tell you...

New Goal

Question

How do we find those magic vectors and magic numbers?

Answer

Well, determinant is not covered yet, so I can't tell you...
At least not why

Determinants

Definition 2

We define a map det which takes input as 2 by 2 matrices and output a real number as follows:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Determinants

Definition 2

We define a map det which takes input as 2 by 2 matrices and output a real number as follows:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Theorem 3

A matrix A is invertible if and only if $\det(A) \neq 0$.

Magic Numbers And Vectors

Magic Numbers And Vectors

1. We want to find λ so that there exists some $0 \neq v$ with $Tv = \lambda v$.

Magic Numbers And Vectors

1. We want to find λ so that there exists some $0 \neq v$ with $Tv = \lambda v$.
2. This is the same as we want to find some v such that $(T - \lambda I)v = 0$

Magic Numbers And Vectors

1. We want to find λ so that there exists some $0 \neq v$ with $Tv = \lambda v$.
2. This is the same as we want to find some v such that $(T - \lambda I)v = 0$
3. This is the same as saying $(T - \lambda I)$ is not invertible, as we are sending something non-zero to zero

Magic Numbers And Vectors

1. We want to find λ so that there exists some $0 \neq v$ with $Tv = \lambda v$.
2. This is the same as we want to find some v such that $(T - \lambda I)v = 0$
3. This is the same as saying $(T - \lambda I)$ is not invertible, as we are sending something non-zero to zero

Magic Number

Thus, to find magic numbers, we just need to solve

$$\det(A - \lambda I) = 0$$

where λ is a variable.

Computation

Well, we see

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

Computation

Well, we see

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

Thus, we see

$$\det(A - \lambda I) = \lambda^2 - \lambda - 1$$

Computation

Well, we see

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

Thus, we see

$$\det(A - \lambda I) = \lambda^2 - \lambda - 1$$

Hence

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

Magic Vectors

Let's try to find the magic vector correspond to $\lambda_1 = \begin{bmatrix} a \\ b \end{bmatrix}$.

Magic Vectors

Let's try to find the magic vector correspond to $\lambda_1 = \begin{bmatrix} a \\ b \end{bmatrix}$.

Assume we don't know what v_1 is, and hence we have

$$\begin{bmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Magic Vectors

Let's try to find the magic vector correspond to $\lambda_1 = \begin{bmatrix} a \\ b \end{bmatrix}$.

Assume we don't know what v_1 is, and hence we have

$$\begin{bmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is exactly the problem of finding solutions to a system of linear equations!

Magic Vectors

Let's try to find the magic vector correspond to $\lambda_1 = \begin{bmatrix} a \\ b \end{bmatrix}$.

Assume we don't know what v_1 is, and hence we have

$$\begin{bmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is exactly the problem of finding solutions to a system of linear equations!

But there are infinitely many solutions to this system of equations.

And we will just pick one generator of the solution space

Magic Vector

In short, one magic vector for λ_1 would be

$$v_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

Magic Vector

In short, one magic vector for λ_1 would be

$$v_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

Similarly, one magic vector for λ_2 would be

$$v_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

Magic Vector

In short, one magic vector for λ_1 would be

$$v_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

Similarly, one magic vector for λ_2 would be

$$v_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

Next, we want to rewrite

$$\begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

in terms of v_1 and v_2

Reap The Fruit

By observation,

$$\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}}(v_1 - v_2)$$

Reap The Fruit

By observation,

$$\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}}(v_1 - v_2)$$

Thus, we see

$$A^{n-1}\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \frac{1}{\sqrt{5}}(\lambda_1^{n-1} v_1 - \lambda_2^{n-1} v_2)$$

Reap The Fruit

By observation,

$$\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}}(v_1 - v_2)$$

Thus, we see

$$A^{n-1}\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \frac{1}{\sqrt{5}}(\lambda_1^{n-1} v_1 - \lambda_2^{n-1} v_2)$$

And so

$$F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$