Contents

Warning

This note contains disturbing typo.

1 What is a Ramen Surface

So you just need to know complex analysis for this course.

In this course we will assume the space we work with is Hausdorff, i.e. $\Delta : X \to X \times X$ is closed immersion. This is equivalent to

$$\forall x, y \in X, \exists U, V \in \mathrm{Op}(X), (U \cap V = \emptyset \wedge x \in U, v \in V)$$

Here for any topological space X, we use $\mathrm{Op}(X)$ to denote the category of open sets, with arrows being inclusion.

Example 1.1

1. $\mathbb{R}^n$ is Hausdorff
2. The affine line with double origin is not separated (i.e. does not have closed diagonal).
3. As topological spaces,Zariski topologies are not Hausdorff (but we can have closed diagonals, hence separated).

Next, when we talk about (topological) surfaces (over a field K), we mean a Hausdorff 2-manifold (over K). Since at the end of the day we work with Ramen surfaces, which are defined over $\mathbb{C}$, in this course we can think of a topological surface as a 1-manifold over $\mathbb{C}$(or 2-manifold over $\mathbb{R}$).

Example 1.2

1. $\mathbb{C} = \mathbb{R}^2$, with one chart $\phi = \mathrm{Id} : \mathbb{C} \to \mathbb{C}$, is a topological surface.
2. If $W \subseteq \mathbb{C}$ is open, then with induced topology and inclusion gives W a structure of a surface.
3. The graph $\Gamma_f = G_f = \{(z, w) \in \mathbb{C}^2 : w = f(z)\}$ of a continuous map $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is a topological surface.
4. $$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} = \underbrace{\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}}_{:=S^2} = [\mathbb{C}^2 \backslash \{0\} / \mathbb{G}_m]$$

   We will put a few words on the last example. First, let us recall the stereographic projection from the north pole N, i.e. $\sigma_N^+ : S^2 \backslash \{(0, 0, 1)\} \to \mathbb{R}^2$ is defined by $(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$. Next, we also have $\sigma_S^+ : S^2 \backslash \{(0, 0, -1)\} \to \mathbb{R}^2$

by $(x,y,z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$. Those two maps gives us the two charts which covers S^2, and hence conclude S^2 is a surface.
On the other hand, note $S^2 \cong \mathbb{C} \cup \{\infty\}$ can be done via $\sigma_N : S^2 \to \mathbb{C} \cup \{\infty\}$, where we lift σ_N^+ by setting $(0,0,1)$ to ∞.

Now also recall complex charts of a surface X, which is just a homeomorphism $\phi : U \subseteq X \to V \subseteq \mathbb{C}$. Given two charts $\phi_i : U_i \to V_i$ with $U_1 \cap U_2 \neq \emptyset$, then we can define its associated transition function (from (ϕ_2, U_2) to (ϕ_1, U_1)) to be $\phi_{21} := \phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2)$. Note this is a complex function.

Definition 1.3

Two charts $\phi_i : U_i \to V_i$ of surface X is ***holomorphically compactible*** if ϕ_{21} and ϕ_{12} are holomorphic (i.e. equivalently ϕ_{12} is biholomorphic).

Definition 1.4

An ***atlas*** on X is a collection $\mathcal{U}$ of charts (ϕ_i, U_i) such that $\bigcup U_i$ covers X. An atlas is ***holomorphic*** if any pair of charts in $\mathcal{U}$ are holomorphically compactible.

Definition 1.5

A ***Ramen surface*** is a topological surface with a complex atlas.

Example 1.6

1. $X = \mathbb{C}$.
2. In general, if X admits an atlas with only one chart $\phi : X \to V \subseteq \mathbb{C}$, then $\mathcal{U} = \{(\phi, X)\}$ is complex, as the compactibility is trivial to check.
3. If $f : U \subseteq \mathbb{C} \to V \subseteq \mathbb{C}$ then Γ_f is a Ramen surface by (2).

Lemma 1.7

If $f : U \subseteq \mathbb{C} \to V \subseteq \mathbb{C}$ is holomorphic bijection, then f^{-1} is holomorphic.

Proof. $(\Rightarrow)$: We will begin by show f is conformal, i.e. $f'(z) \neq 0$ for all $z \in U$. Now let $z_0 \in U$ and set $g = f - f(z_0)$. Then $g(z_0) = 0$, g is holomorphic bijection and $f' = g'$ on U.

Since g is holomorphic and $g(z_0) = 0$, $g(z) = (z - z_0)^m h(z)$ with h holomorphic and $h(z_0) \neq 0$. But $g(z)$ is a bijection, and thus $m = 1$ (otherwise $g'(z_0) = 0$). Hence $g(z) = (z - z_0)h(z)$ and so $g'(z) = h(z) + (z - z_0)h'(z)$, and so

$$f'(z_0) = h(z_0) \neq 0$$

as desired. This immediately implies f^{-1} is holomorphic, i.e. $f^{-1}(w) = f^{-1}(w, \overline{w})$.

Then $\frac{\partial f^{-1}}{\partial \overline{w}} = 0$, as $z = f^{-1}(f(z))$. Since f is holomorphic, $\frac{\partial f}{\partial z} = 0$. Then

$$0 = \frac{\partial}{\partial \overline{z}}(z) = \frac{\partial}{\partial \overline{z}}(f^{-1}(f(z))) = \frac{\partial f^{-1}}{\partial w}\frac{\partial w}{\partial \overline{z}} + \frac{\partial f^{-1}}{\partial \overline{w}}\frac{\partial \overline{w}}{\partial \overline{z}}$$

Now note

$$\frac{\partial w}{\partial \overline{z}} = 0, \quad \text{and } \frac{\partial \overline{z}}{\partial \overline{z}} = \frac{\partial \overline{f}}{\partial \overline{z}} = \overline{\frac{\partial f}{\partial z}}$$

This implies

$$\frac{\partial f^{-1}}{\partial \overline{w}} \circ \overline{f'(z)} = 0 \Rightarrow \frac{\partial f^{-1}}{\partial \overline{w}} = 0 \text{ since } f'(z) \neq 0$$

This shows f' is holomorphic.

As a consequence of this lemma, we see to check for complex atlas, we only need to check $\phi_i \circ \phi_j^{-1}$ is holomorphic for all $i, j \in I$.

Remark 1.8

If a topological surface admits a complex atlas, then its topology is second countable (and hence paracompact).

Let us now keep talking about the example $S^2 = \mathbb{C} \cup \{\infty\}$. The topology for $\mathbb{C} \cup \{\infty\} =: \mathbb{P}^1$ is given by $U \subseteq \mathbb{P}^1$ open iff:

1. $U \subseteq \mathbb{C}$ is open (with standard metric topology), or
2. $U = (\mathbb{C} \backslash K) \cup \{\infty\}$ for some $K \subseteq \mathbb{C}$ compact.

One can show this is Hausdorff, as one can show (left as exercise). Next, we can put a complex atlas on $\mathbb{P}^1$, and the standard is given by $\phi_1 : \mathbb{C} \to \mathbb{C}$ and $\phi_2 : \mathbb{C}^* \cup \{\infty\} \to \mathbb{C}$, where

$$\phi_1(z) = z, \quad \phi_2(z) = \frac{1}{z}$$

where we set $\frac{1}{\infty} = 0$. One checks ϕ_1, ϕ_2 are homeomorphism with inverses, and it remains to show ϕ_1, ϕ_2 are holomorphically compactible. It suffices to check $\phi_1 \circ \phi_2^{-1}$ is holomorphic by the above lemma. However, on the overlap (i.e. $\mathbb{C}^* \to \mathbb{C}^*$), this function is just $\frac{1}{z}$ and hence holomorphic.

Definition 1.9

Two complex atlases $\mathcal{U}$ and $\mathcal{V}$ are called ***analytically equivalent*** if every chart of $\mathcal{U}$ is holomorphically compactible with every chart in $\mathcal{V}$.

This is an equivalence relation, since composition of biholomorphic functions is still biholomorphic.

Definition 1.10

A ***complex structure*** Σ on a topological surface X is an equivalence class of analytically equivalent atlases.

Its clear that a complex atlas on X determines a complex structure, and conversely any complex structure is determined by a unique complex atlas, i.e. the maximal complex atlas.

Thus, we have the following definition:

Definition 1.11

A ***Ramen surface*** is a pair (X, Σ) where X is a topological surface and Σ is a complex structure.

Let us finish today's lecture with one last example, the complex torus.

Let $\Gamma = \text{span}_{\mathbb{Z}}(\omega_1, \omega_2)$, where $\omega_1, \omega_2 \in \mathbb{C}$ linearly independent (viewed as $\mathbb{R}^2$) over $\mathbb{R}$. Then, the complex torus associated with Γ is the quotient space $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ (with quotient topology). In particular, we see:

1. X is connected since $\mathbb{C}$ is.
2. X is compact since its the image of a compact set (i.e. $\text{Conv}(0, \omega_1, \omega_2, \omega_1 + \omega_2)$) in $\mathbb{C}$ under π.
3. $X \cong S_1 \times S_1$ homeomorphically.
4. π is open. To see this one, it suffices to show $\pi^{-1}(\pi(U))$ is open in $\mathbb{C}$ for all $U \subseteq \mathbb{C}$. However, $\pi^{-1}(\pi(U)) = \bigcup_{\omega \in \Gamma}(U + \omega)$, which is indeed open.
5. X is Hausdorff (we can see this from the fact that the set of orbit equivalence relation $\{(x, y) : [x] \sim [y] \in X\} = \{(x, y) : x = y + m\omega_1 + n\omega_2, n, m \in \mathbb{Z}\}$ in $\mathbb{C} \times \mathbb{C}$ is closed).

In the next lecture we will show X admits a complex atlas.

Today we will finish the example on torus.

Recall above we have $X = \mathbb{C}/\Gamma$ with $\Gamma = \text{span}_{\mathbb{Z}}(\omega_1, \omega_2)$, where $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent as $\mathbb{R}$-vectors.

In order to show X is a Ramen surface, we first need to show X is a surface. In the above, recall $\pi : \mathbb{C} \to X$ is an open map (since the action is homeomorphism).

First we need to show it is Hausdorff. For this, we will show

$$E := \{(x, y) \in \mathbb{C} \times \mathbb{C} : [x] \sim [y]\}$$

is a closed subset in $\mathbb{C} \times \mathbb{C}$. To see this, just take a convergent sequence $\{(x_n, y_n)\}_{n \geq 1}$ such that $y_n = x_n + \omega_n$ for some $\omega_n \in \Gamma$. This means $\lim x_n = x$ and $\lim y_n = \lim(x_n + \omega_n) = y$, but if you think about it we see the choice of ω_n has to stablize to some ω_0 at some point (otherwise $x_n + \omega_n$) will not converge. That is, $\lim(x_n, y_n)$ lies in E as well, shows it is closed. Since E is closed, the diagonal of $X \to X \times X$ is closed, hence Hausdorff.

Now it remains to give a complex atlas to X. For all $p \in X$, choose $\tilde{p} \in P$ be the unique representative of p in the fundamental parallelogram and V_p an open neighbourhood of p small enough so $\pi|_V$ is one-to-one. Now just set $U_p = \pi(V_p)$ and clearly U_p is open in X with $p \in U_p$. Set ϕ_p be defined as $(\pi|_{V_p})^{-1} : U_p \to V_p$, then this gives a chart around p.

It remains to check those are holomorphically compactible.

Let $p, q \in X$ and consider $\psi = \phi_p \circ \phi_q^{-1}$, where recall $\phi_p = (\pi|_{U_p})^{-1}$ and $\phi_q^{-1} = \pi$. A computation shows

$$\pi(\psi(z)) = \pi \circ \phi_p \circ \phi_q^{-1}(z) = \pi(z)$$

Thus we see $\psi(z) \sim z$ under our equivalence relation, and thus $\psi(z) - z \in \Gamma \cong \mathbb{Z}^2$, which is a discrete group. However, $\psi(z) - z$ is continuous, and thus it must be constant, i.e. it is holomorphic.

The last example we do is algebraic curves.

Let $P(z, w)$ be a non-constant polynomial in complex variables z, w. Then we define

$$C := \{(z, w) \in \mathbb{C}^2 : P(z, w) = 0\}$$

We say C is smooth at (z_0, w_0) if $\nabla P(z_0, w_0) := (\frac{\partial P}{\partial z}(z_0, w_0), \frac{\partial P}{\partial w}(z_0, w_0))$ is non-zero, and otherwise its singular at the point.

Example 1.12

1. Let $P = w - z^2$, then C is smooth
2. Let $P = w^2 - z^2$, then C is non-singular at $(0, 0)$

The point is, all algebraic curves are Ramen surface, away from the singular points (e.g. $P = w^2 - z^2$ then $C \backslash \{(0, 0)\}$ is a Ramen surface).

Proposition 1.13

Let $S = C \setminus \mathrm{Sing}$ (here Sing is the set of singular points of C). Then S admits a natural complex structure, making it into a Ramen surface.

The above theorem is a direct consequence of the implicit function theorem. Recall implicit function theorem says if (z_0, w_0) is a point on C s.t. $\frac{\partial P}{\partial w}(z_0, w_0) \neq 0$. Then there is a disc D_1 centered at z_0 in $\mathbb{C}$, D_2 centered at $w_0 \in \mathbb{C}$, a holomorphic map $\phi : D_1 \to D_2$, with $\phi(z_0) = w_0$ such that

$$C \cap (D_1 \times D_2) = \{(z, \phi(z)) : z \in D_1\}$$

Proof. By assumption, $(z_0, w_0) \in S$ means $\nabla P(z_0, w_0)$ is non-zero. Thus we get a chart of S that is locally just the graph of a holomorphic function.

Definition 1.14

Let X be a Ramen surface and $Y \subseteq X$ open, then we define $\mathscr{O}(Y)$ as the set of holomorphic functions $f : Y \to \mathbb{C}$.

In the above, $f : Y \to \mathbb{C}$ is holomorphic if for all charts $(\phi : U \to V)$, we have $f \circ \phi^{-1} : \phi(U \cap Y) \subseteq V \to \mathbb{C}$ is holomorphic.

Remark 1.15

1. With the obvious restriction, we see $\mathscr{O}$ is a sheaf (this means $\mathscr{O} : \mathrm{Op}(X) \to$ (**Rings**) is a functor, plus some equalizer condition) valued in rings on X.
2. In fact, $\mathscr{O}_X$ is a $\mathbb{C}$-algebra.
3. As corollary of (1), to check $f \in \mathscr{O}(Y)$, it suffices to show $f|_{U_i} \in \mathscr{O}(U_i)$ for an open cover $\{U_i\}$ of Y.

2 Holomorphic Mapping and Meromorphic Functions

Let us begin with examples.

Example 2.1

We classify $f : \mathbb{P}^1 \to \mathbb{C}$.

In this case, $\mathbb{P}^1$ admits a chart $U_1 = \mathbb{C}$ and $U_2 = \mathbb{C}^* \cup \{\infty\}$, where $\phi_1 = \mathrm{Id}$ and $\phi_2 = \frac{1}{z}$.

Then we see $f \circ \phi_1^{-1} = f : \mathbb{C} \to \mathbb{C}$ and $f \circ \phi_2^{-1} = f(1/z) : \mathbb{C} \to \mathbb{C}$ must both be holomorphic.

Since f is holomorphic, we can do Taylor expansion to conclude $f(z) = \sum_{h=0}^{\infty} a_n z^n$. However, we see $f(1/z)$ must also be holomorphic, and $f(z) = f(1/z)$ for all $z \in \mathbb{C}^*$ (the overlap of the two charts). This suggests $f(z)$ is a constant a_0 for all $z \in \mathbb{C}^*$, but f is continuous, hence $f(0)$ must also equal a_0, i.e. f is just a constant.

This concludes $\mathscr{O}(\mathbb{P}^1) = \mathbb{C}$.

There is nothing special about $\mathbb{P}^1$ here, as all we need is this complex manifold to be compact, then the global sections must be constant.

Theorem 2.2: Riemann's Removable Singularity

Let U be an open subset of Ramen surface X and $a \in U$. Suppose $f \in \mathcal{O}(U \backslash \{a\})$ is bounded on a neighbourhood of a, then f can uniquely extended to $f^* \in \mathcal{O}(U)$.

Proof. Apply the complex analysis version of this theorem.

Definition 2.3

Let X, Y be Ramen surfaces, then a continuous mapping $f : X \to Y$ is ***holomorphic*** if for all pairs of charts ψ_1, ψ_2, the composition $\psi_2 \circ f \circ \psi_1^{-1}$ is holomorphic as complex function.

Then, we say f is biholomorphic if it is a bijection and both f and f^{-1} are holomorphic. Finally, X, Y are isomorphic if there is biholomorphic mapping $f : X \to Y$.

Remark 2.4

The definition of holomorphic mapping is equivalent to $f^* : \mathcal{O}_Y \to f^*\mathcal{O}_X$ is well-defined. Here $f^*\mathcal{O}_X$ is a sheaf on Y defined by $f^*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$.

Example 2.5

1. Any holomorphic function $f : \mathcal{O}(X)$ is a holomorphic mapping $f : X \to \mathbb{C}$
2. Composition of holomorphic mappings are holomorphic
3. Let $X = \mathbb{C}/\Gamma$ and $Y = \mathbb{C}/\Gamma'$ be two complex tori, then $X \cong Y$ iff $\Gamma = \Gamma'$.
4. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be given by
$$z \mapsto \frac{az+b}{cz+d}$$
with $ad - bc = 1$ is an automorphism of $\mathbb{P}^1$. Note in this case, $-d/c \mapsto \infty$, $-b/a \mapsto 0$ and $\infty \mapsto a/c$. The inverse is given by
$$z \mapsto \frac{dz-b}{-cz+a}$$

Theorem 2.6: Identity Theorem

Suppose X, Y are Ramen surfaces and $f_1, f_2 : X \to Y$ be two holomorphic mapping coincide on a set $A \subseteq X$ with a limit point $a \in X$. Then $f_1 = f_2$.

Before we prove this result, let us first recall the notion of isolated point and limit point for topological spaces. For $S \subseteq X$. We say x is isolated point if $x \in S$ and there is open neighbourhood of x such that $S \cap (U \backslash \{x\}) = \emptyset$. We say x is limit point if for all open neighbourhood of x, $S \cap (U \backslash \{x\}) \neq \emptyset$. Note here limit points need not be in S.

Proof. Set $B = \{x \in X : f(x) = g(x)\}$. This is the equalizer of f and g and we see $A \subseteq B$. Now let

$$G = \{x \in X : f|_{W_x} = g|_{W_x} \text{ for some open neighbourhood } W_x\}$$

and clearly $G \subseteq B$. We will show G is open, closed and non-empty, then $G = X$ and thus $B = X$.

To show G is open, let $x \in G$. Then we can find open neighbourhood W_x of x such that $f|_{W_x} = g|_{W_x}$. But W_x is open neighbourhood for all $x' \in W_x$, i.e. $W_x \subseteq G$. Hence G is open.

Next we claim x is a limit point of B then $x \in G$. This would imply G contains all its limit points and non-empty, since $A \subseteq B$ contains a limit point. This would conclude our proof.

It remains to prove the claim.

First we show if x is a limit point of B then $x \in B$.

Let $\phi : U \to V$ be a chart around x, where x is a limit point of B. Since $f(x) \in Y$, we can pick a chart $\phi' : U' \to V'$ around $f(x)$. Note we can choose ϕ so that $f(U) \subseteq U'$. Then $\tilde{f} := \phi' \circ f \circ \phi^{-1}$ is a holomorphic function from $\phi(U) \subseteq \mathbb{C}$ to $\phi'(U') \subseteq \mathbb{C}$. Since x is a limit point of B, $\phi(x)$ is a limit point of $\phi(U \cap B)$ in $\mathbb{C}$. Thus we can find a sequence $y_n \in \phi(U \cap B)$ with $\lim y_n = \phi(x)$. By definition of y_n, we see $y_n = \phi(x_n)$ for some $x_n \in U \cap B$. This shows $\lim \phi(x_n) = \phi(x)$. By continuity of $\tilde{f}$, we see

$$\tilde{f}(\phi(x_n)) \to \tilde{f}(\phi(x))$$

Similarly, $\tilde{g}(\phi(x_n)) \to g(\phi(x))$ where $\tilde{g} = \phi' \circ g \circ \phi^{-1}$. However, since $x_n \in U \cap B$, it means f and g agrees on x_n, i.e. we get

$$\tilde{f}(\phi(x_n)) = \phi'(f(\phi^{-1}(\phi(x_n)))) = \phi'(f(x_n)) = \phi'(g(x_n)) = \tilde{g}(\phi(x_n))$$

Thus we see $\phi'(f(x)) = \phi'(g(x))$ by continuity, and hence $f(x) = g(x)$ as ϕ' is a homeomorphism, i.e. $x \in B$.

We also want to show there exists open neighbourhood W_x of x where $f|_{W_x} = g|_{W_x}$. Note $\tilde{f}, \tilde{g}$ are holomorphic functions such that $\tilde{f} = \tilde{g}$ on $\phi(U \cap B)$ and $\phi(U \cap B)$ has a limit point. Thus $\tilde{f} = \tilde{g}$ by the Identity theorem on $\mathbb{C}$, i.e. $x \in G$.

As a consequence of the Identity theorem, we have:

Corollary 2.6.1

Let $f : X \to Y$ be a holomorphic mapping. If f is not constant then $f^{-1}(y)$ will be a set of isolated points, for all $y \in f(X) \subseteq Y$.

Proof. Let $y_0 \in f(X)$ and $S = f^{-1}(y_0)$. Then let $g : X \to Y$ is defiend by $x \mapsto y_0$. If S contains a limit point, then by the above, we see $f = g$ is constant, a contradiction. Hence S must only contain isolated points.

Corollary 2.6.2

Let $f : X \to Y$ be a holomorphic mapping between Ramen surfaces. If X is compact then $|f^{-1}(y)| < \infty$.

Proof. This is because $f^{-1}(y)$ is closed subset of X and thus also compact. Now, for all $x \in f^{-1}(y)$, we can find open neighbourhood U_x of x with $f^{-1}(y) \cap (U_x \setminus \{x\}) = \emptyset$ (as $f^{-1}(y)$ contains only isolated points). Thus $\bigcup_{x \in f^{-1}(y)} U_x$ is an open cover of $f^{-1}(y)$ and hence we can find a finite subcover, i.e. $f^{-1}(y)$ is finite set.

Theorem 2.7: Local Behaviour of Holomorphic Mapping

Let $f : X \to Y$ be a non-constant holomorphic mapping, $a_0 \in X$ and $b_0 = f(a_0)$. Then there exists integer $k \geq 1$ and charts $\phi : U \to V$ on X and $\psi : U' \to V'$ on Y, with following properties:

1. *$a_0 \in U$, $\phi(a_0) = 0$, $b_0 \in U'$ and $\psi(b_0) = 0$*
2. *$f(U) \subseteq U'$*
3. *The map $F := \psi \circ f \circ \phi^{-1}$ is given by $F(z) = z^k$ for all $z \in V$ where $k \geq 1$*

The idea of the above proof is simple. Start with any chart, say $\phi : U_1 \to V_1$ and $\psi : U_2 \to V_2$, of X and Y, respectively, with ϕ around a_0 and ψ around b_0. WLOG we may assume (by translation) $\phi(a_0) = 0$ and $\psi(b_0) = 0$. But then $\psi \circ f \circ \phi^{-1}$ is holomorphic and a computation shows $F(0) = 0$. Now we see $F(z) = z^k g(z)$ for some $k \geq 1$ and g holomorphic at 0 with $g(0) \neq 0$. Since g is holomorphic and non-zero at 0, its kth root exists, i.e. we can find holomorphic h on open neighbourhood near 0, such that $h^k = g$. Now just set our new variable t be $z \cdot h$, we see $F = t^k$ as desired.

In particular, the k in the theorem above is called the multiplicity of f at a_0.

Also, throughout, it is safe to assume the mappings between Ramen surfaces are non-constant.

Here are some global results.

Theorem 2.8

Let X, Y be connected Ramen surfaces and $f : X \to Y$ be a non-constant holomorphic mapping. Then:

1. *f is open*
2. *if $Y = \mathbb{C}$, then $|f|$ does attain its maximum on X*

The two statements in the above theorem have their own names: open mapping theorem, and maximum modulus principle.

Corollary 2.8.1

Suppose X, Y are connected Ramen surfaces, with X compact. If $f : X \to Y$ is non-constant holomorphic mapping, then Y is compact and f is surjective.

Proof. Since f is continuous, $f(X)$ is compact. It remains to check $f(X) = Y$. Since Y is connected, it is enough to check $f(X)$ is open, closed and $\neq \emptyset$. However, it is clearly closed as X is compact and non-empty. In addition, since f is open, $f(X)$ is open.

Corollary 2.8.2

Every holomorphic function $f : X \to \mathbb{C}$ on compact Ramen surface is constant.

Proof. X is compact and $\mathbb{C}$ is connected. By above result, it cannot be non-constant, as that would imply $\mathbb{C}$ is compact.

Example 2.9

1. $\mathscr{O}(\mathbb{P}^1) = \mathbb{C}$
2. $\mathscr{O}(\mathbb{C}/\Gamma) = \mathbb{C}$. We can see this directly: say $f : \mathbb{C}/\Gamma \to \mathbb{C}$ is the same as some $\tilde{f} : \mathbb{C} \to \mathbb{C}$ that's constant on the orbit. That is, $\tilde{f}(z + m\omega_1 + n\omega_2) = f(z)$ for all $n, m \in \mathbb{Z}$, i.e. they are elliptic functions, i.e. doubly-periodic. Thus we see holomorphic doubly-periodic functions are constants.

Recall: let $D \subseteq \mathbb{C}$ be domain and $a_0 \in D$. Consider $f \in \mathscr{O}(D \backslash \{a_0\})$. Then, a_0 is an isolated singularity of f and must be of the form of the following three types:

1. If $\lim_{z \to a_0} f(z)$ exists, then a_0 is a removable singularity

2. If $\lim_{z\to a_0} |f(z)| = \infty$, then a_0 is a pole of f
3. If $\lim_{z\to a_0} f(z)$ does not exist and $\lim_{z\to a_0} |f(z)| \neq \infty$, then a_0 is an essential singularity

Then a complex function $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is called meromorphic if it is holomorphic on D except on a set of isolated points where it has poles.

Example 2.10

1. $f(z) = \frac{\sin(z)}{z}$ has removable singularity at $z = 0$
2. $f(z) = \frac{1}{z}$ has pole at $z = 0$
3. $f(z) = e^{1/z}$ has essential singularity at $z = 0$

Definition 2.11

Let X be a Ramen surface and Y an open subset of X. Then a ***meromorphic*** function on Y is a holomorphic function $f : Y' \to \mathbb{C}$, where $Y' \subseteq Y$ and such that:

1. $Y \backslash Y'$ only contains isolated points
2. for all points in $Y \backslash Y'$, one has $\lim_{x\to p} |f(x)| = \infty$

The points in $Y \backslash Y'$ are called the poles of f.

The set of all meromorphic functions on Y is denoted by $\mathcal{U}(Y)$.

Example 2.12

1. Consider $f : \mathbb{P}^1 \to \mathbb{C}$ given by
$$z \mapsto \frac{az+b}{cz+d}$$
where $ad - bc \neq 0$. This is holomorphic away from $z = -d/c$ and has a pole at $z = -d/c$.
2. Let $n \geq 1$ and $f(z) = \sum_{i=0}^{n} a_i z^i$. Then $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and thus we can extend this to $f : \mathbb{P}^1 \to \mathbb{C}$ by sending ∞ to ∞.

Theorem 2.13

Suppose X is a Ramen surface and $f \in \mathcal{U}(X)$. For each pole p of f, we define $\hat{f}(p) = \infty$ and $\hat{f}(x) = f(x)$ otherwise. Then $\hat{f} : X \to \mathbb{P}^1$ is holomorphic.

Proof. Let P (or you can denote this by $\mathrm{div}(f)$) be the set of poles of f. Then $f : X \backslash P \to \mathbb{C}$ is holomorphic and it induces the mapping $\hat{f} : X \to \mathbb{P}^1$ as above. Note $\hat{f}$ is a continuous function on X, as it is continuous on $X \backslash P$ and

$$\lim_{x\to x_0} \hat{f}(x) = \hat{f}(\lim_{x\to x_0} x) = \infty$$

for all $x_0 \in P$.

It remains to show $\hat{f}$ is holomorphic. To that end, we apply removable singularity theorem. Let $x_0 \in P$, we can find an open neighbourhood U of x_0 so $U \cap P = \{x_0\}$. We have

$$f(x_0) = \infty \in \mathbb{P}^1$$

and thus pick the chart $\psi(z) = \frac{1}{z}$ on $\mathbb{C}^* \cup \{\infty\}$. We check in this case f is bounded on this chart. Let $\epsilon > 0$ be small enough so $\psi^{-1}(B_\epsilon(0)) \subseteq f(U)$. Set

$$W = (\psi \circ \hat{f})^{-1}(B_\epsilon(0)) \subseteq U$$

Then W is open and $x_0 \in W$. Then $\psi \circ \hat{f}|_W : W \to B_\epsilon(0)$ is bounded with $\psi \circ \hat{f}|_W$ is holomorphic on $W \backslash \{x_0\}$, i.e. $\psi \circ \hat{f}$ extends to holomorphic function on W by Riemann's removable singularity theorem.

3 Branched and Unbranched Coverings

Recall if X is topological space, then:

1. if X compact then any closed subset is compact
2. if X is Hausdorff then any compact subset is closed
3. if X Hausdorff and A discrete (i.e. only contains isolated points). If X is compact then A is finite.

Definition 3.1

A continuous map $p : X \to Y$ is:

1. ***discrete*** if $p^{-1}(y)$ is discrete for all y
2. ***finite*** if $p^{-1}(y)$ is finite for all y

In particular, by the above facts if X compact Hausdorff, then discrete maps are finite.

Theorem 3.2

Let X, Y be Ramen surfaces, and $f : X \to Y$ is non-constant holomorphic map. Then f is discrete. In particular, if X is compact, then f is finite.

Proof. Clear.

Its natural to ask, if X is not compact, then what condition do we need to make sure f is finite. Well, note compact discrete sets are finite.

Definition 3.3

A holomorphic map $f : X \to Y$ between Ramen surfaces is ***proper*** if $f^{-1}(K)$ is compact for all compact $K \subseteq Y$.

Thus, we see if f is proper, then $f^{-1}(y)$ are compact, i.e. its compact and discrete and hence finite. Thus we see proper maps are finite. Let us record this as a theorem.

Theorem 3.4

Let X, Y be Ramen surfaces and $f : X \to Y$ is non-constant proper map. Then f is finite.

Now it is natural to ask the following:

Question

If f is finite, then is it proper?

Turns out it is true if f is closed.

Example 3.5

1. If X is compact Ramen, then $f : X \to Y$ is proper
2. Let $f : \mathbb{C} \to \mathbb{C}$ be constant, then its not proper
3. Let $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ be the projection, then its not proper

Next, we will show if f is proper, then $f^{-1}(y)$ all has the same number of elements.

Definition 3.6

Let X, Y be Ramen surfaces and $f : X \to Y$ a non-constant holomorphic map. A point $x \in X$ is called a ***branched point*** (or ***ramification point***) if there does not exist open neighbourhood U of x such that $f|_U$ is injective.

Otherwise we say its unbranched. We say f is unbranched if it does not have branched points.

Example 3.7

1. $f:\mathbb{C}\to\mathbb{C}$ given by $z\mapsto z^k$ for $k\geq 2$ is branched at 0.
2. $f:\mathbb{C}\to\mathbb{C}^*$ given by $z\mapsto e^z$ is unbranched.
3. More generally, if $f:X\to Y$ is a non-constant holomorhpic map that locally looks like $F(z)=z^k$, then its unbranched iff $k=1$
4. $\pi:\mathbb{C}\to\mathbb{C}/\Gamma$ is unbranched for any lattice

Here are some facts about non-constant holomorphic maps $f:X\to Y$ between Ramen surfaces:

1. its open
2. locally it looks like $z\mapsto z^k$
3. f proper implies f finite

We will see the following:

1. if f is unbranched then:
   (a) its local homeomorphism
   (b) if f proper then all fibers have the same size
2. if f is branched then:
   (a) the set A of all branched points of f is closed and discrete
   (b) if f proper, then $f(A)$ is closed and discrete

Definition 3.8

Let $p:X\to Y$ be holomorphic non-constant map between Ramen surfaces. Then:

1. p is a ***local homeomorphism*** if for all $x\in X$, $\exists$ open neighbourhood U of x in X so $p|_U:U\to p(U)$ is a homeomorphism
2. p is a ***covering map*** if for all $y\in Y$, $\exists$ open neighbourhood V of y in Y so

$$p^{-1}(V)=\bigcup_{j\in J}U_j$$

   with each U_j open, disjoint, and $p|_{U_j}$ is a homeomorphism.

Its not hard to see:

1. every covering map is local homeomorphism
2. not every local homeomorphism is a covering map

Example 3.9

Let $U\subsetneq X$ be open subset and i inclusion. Then i is local homeomorphism but not covering map. In addition, surjective local homeomorphism can fail to be covering map as well. For exmaple, consider $p:(0,2)\subseteq\mathbb{R}\to S^1\subseteq\mathbb{C}$ with $t\mapsto e^{2\pi it}$. Then this is not a covering map.

On the other hand, if you take $p:\mathbb{R}\to S^1$ by $t\mapsto e^{2\pi it}$ then it is.

Example 3.10

1. $p : \mathbb{C}^* \to \mathbb{C}^*$ by $t \mapsto t^k$ is a covering map for $k \geq 1$. To see this, let $a, b \in \mathbb{C}^*$ so $a^k = b$. Let $w = e^{2\pi i/k}$ be the primitive kth roots of unit so $w^j \neq 1$, $0 < j < k$, and $w^k = 1$. Then

$$p^{-1}(b) = \{a, aw, aw^2, ..., aw^{k-1}\}$$

and thus $p'(z) = kz^{k-1} \neq 0$ for all $z \in \mathbb{C}^*$, i.e. p is locally invertible. Thus we can choose U_0 of a so that $p|_{U_0}$ is homeomorphism, and we can choose U_0 small enough so $aw^j \notin U_0$ for $0 < j < k$. Now set $U_j = w^j \cdot U_0$ for $0 < j < k$. Those will be all disjoint with the same image $p(U_0)$
2. $\exp : \mathbb{C} \to \mathbb{C}^*$ given by $z \mapsto \exp(z)$ is a covering map.
3. $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is a covering map.

Theorem 3.11

Let $p : X \to Y$ be a covering map. Then, if X is connected, p is surjective and $\forall y_0, y_1 \in Y$, $|p^{-1}(y_0)| = |p^{-1}(y_1)|$.

Proof. Let $y_0 \in Y$. Since p is a covering map, we can find $V \subseteq Y$ so $p^{-1}(V) = \coprod_{j\in J} U_j$ with each $U_j \cong V$. Then $|p^{-1}(y_0)| = |J|$. In fact, $|p^{-1}(y)| = |J|$ for all $y \in V$. In particular we see

$$p^{-1}(y) = \coprod_{j\in J} p^{-1}(y) \cap U_j$$

THus, for all $x \in p^{-1}(y)$, we see $x \in \coprod_{j\in J} U_j$ implies $x \in U_j$ for some unique j. Hence $|p^{-1}(y)| \leq |J|$. To check $|J| \leq |p^{-1}(y)|$, we need to show for all $j \in J$, we can find $x \in U_j$ so $p(x) = y$. But each $p|_{U_j}$ is homeomorphism, hence surjective, i.e. $|J| = |p^{-1}(y)|$.

Now let $A = \{y \in Y : |p^{-1}(y)| = |J|\}$. We want to show $A = Y$.

We first show A is open. Since p is a covering map, we can find open neighbourhood $W \subseteq Y$ such that $p^{-1}(w) = \bigcup_{l\in\tilde{J}} \tilde{U}_l$ with $\tilde{U}_l$ oepn disjoint and $p|_{\tilde{U}_l}$ are homeomorphisms. Then, for all $z \in W$, $|p^{-1}(z)| = |\tilde{J}|$. But $y \in W$ so

$$|\tilde{J}| = |p^{-1}(y)| = |J|$$

This implies $|p^{-1}(z)| = |J|$ for all $z \in W$, i.e. $W \subseteq A$, i.e. A is open.

It remains to show $A = Y$. In this case, let $A_k = \{y \in Y : |p^{-1}(y)| = k\}$, then A_k are open, disjoint and covers Y. Since each A_k is open and disjoint, we must have $Y = A_{k_0}$ for some k_0 by connectedness of Y. Thus $Y = A_{k_0}$ for some k_0. Note $X \neq \emptyset$, so $p(X) \neq \emptyset$ and thus $p^{-1}(y_0) \neq \emptyset$ for some $y_0 \in p(X)$. Then for all $y \in Y$, $|p^{-1}(y)| = |p^{-1}(y_0)| \neq 0$. THis shows p is surjective.

Definition 3.12

Let p be a covering map. Then the ***number of sheets*** of p is $|p^{-1}(y)|$.

This number may be finite or infinite.

Example 3.13

1. $p : \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t}$ has an infinite number of sheets.
2. $z \mapsto z^k$ has k sheets.

Theorem 3.14

Let X, Y be Ramen surfaces with f non-constant holomorphic map. Then:

1. *f is unbranched iff its a local homeomorphism*
2. *f is proper and Unbranched then it is a covering map*