Contents

1 Categories

Category is cool.

Definition 1.1

A ***category*** $\mathcal{C}$ consists of:

1. a class of ***objects*** $\mathrm{Obj}(\mathcal{C})$
2. a class of ***morphisms*** $\mathrm{Hom}_{\mathcal{C}}(A,B) = \mathrm{Hom}(A,B)$ for every pair of $A,B \in \mathrm{Obj}(\mathcal{C})$

such that:

1. there is a map $\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C)$ called composition, and we write this as $(f,g) \mapsto f \circ g \in \mathrm{Hom}(A,C)$
2. for $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have $(f \circ g) \circ h = f \circ (g \circ h)$
3. there is an identity map $\mathrm{Id} \in \mathrm{Hom}(A,A)$ such that $\mathrm{Id}_A \circ f = f$ and $g \circ \mathrm{Id}_B = g$ for all $A,B \in \mathrm{Obj}(\mathcal{C})$

Definition 1.2

Let $\mathcal{C}$ be a category, $A,B \in \mathcal{C}$. Then we say A and B are ***isomorphic***, and write $A \cong B$, if there is a morphism $f : A \to B$ such that there is a morphism $g : B \to A$, with $f \circ g = \mathrm{Id}_B$ and $g \circ f = \mathrm{Id}_A$.

In the future we will use $\mathcal{C}$ to denote the object as well.

We remark that there are some subtly points to be made about the foundation. Namely, here when we say class we mean a collection that can be a set or can be bigger than sets (e.g. "the set of all sets" is a proper class).

Definition 1.3

A ***locally small category*** $\mathcal{C}$ is a category where $\mathrm{Hom}(A,B)$ is a set for every pair of $A,B \in \mathcal{C}$.

All the things we learned in undergrad and most grad courses are categories.

Example 1.4

1. We have the category of sets (**Set**), the morphisms are set maps
2. The category of groups (**Grp**), the morphisms are group homomorphisms
3. The category of topological spaces (**Top**), morphisms are continuous maps
4. The category of pointed topological spaces (**Top***), here the objects are pointed topological spaces, i.e. a pair (X,x) with $x \in X$, and morphisms are continuous maps $f : (X,x) \to (Y,y)$ with $f(x) = y$
5. The category of abelian groups (**Ab**)

6. The category of rings (**Ring**)
7. The category of left R-modules (R-**Mod**), which we will use this to denote the category of left modules
8. The category of right R-modules (**Mod**-R).
9. Let X be a topological space, then we define $\text{Op}(X)$ as the category of open sets of X, with morphism being $U \to V$ if and only if $U \subseteq V$
10. The category of Lie algebra over k, (**Lie**$_k$), where objects are Lie algebras. That is, a Lie algebra is a k-vector space V, together with a Lie bracket $[,]: V \times V \to V$ such that:
    - $[,]$ is bilinear, i.e. $[\alpha v_1 + v_2, w] = \alpha[v_1, w] + [v_2, w]$
    - $[v, w] = -[w, v]$
    - $[[v, w], u] = [[v, u], w] + [v, [w, u]]$

    The morphisms are of course k-linear maps that preserves the Lie bracket, i.e. $\phi([v, w]) = [\phi(v), \phi(w)]$. Here is a basic examples of Lie algebra: take R be a ring (need not be commutative), $k \hookrightarrow R$, with $k \subseteq Z(R) = \{z \in R : zx = xz \text{ for all } x \in R\}$, and $[r, s] = rs - sr$.

Definition 1.5

A ***groupoid*** is a small category where all morphisms are invertible.

Immediately we see a group is the same as a groupoid with the object being a singleton, i.e. $\text{Obj}(G) = \{*\}$.

Example 1.6

Let $\mathcal{C}$ be a groupoid such that $|\text{Hom}(A, B)| \leq 1$ for $A, B \in \mathcal{C}$. Then this is the same as a equivalence relation on the set $\text{Obj}(\mathcal{C})$.

A functor is just a morphism between categories:

Definition 1.7

Let $\mathcal{C}, \mathcal{D}$ be two categories. A ***functor*** $F : \mathcal{C} \to \mathcal{D}$ consists of:

1. $F : \text{Obj}(\mathcal{C}) \to \text{Obj}(\mathcal{D})$, which we write as $F(A)$ or FA.
2. $F : \text{Hom}(A, B) \to \text{Hom}(FA, FB)$ for all $A, B \in \mathcal{C}$, which we write as $F(f)$ or Ff for $f : A \to B$.

such that:

1. $F(\text{Id}_C) = \text{Id}_{F(C)}$
2. $F(f \circ g) = F(f) \circ F(g)$

Example 1.8

1. The forgetful functor $G : (\textbf{Ab}) \to (\textbf{Grp})$ which sends the abelian group G to the group G.
2. The abelianization functor $A : (\textbf{Grp}) \to (\textbf{Ab})$ which sends the group H to

the abelian group $H/[H,H]$, where recall $[H,H]$ is the commutator of H.
3. The forgetful functor $G : (\mathbf{Grp}) \to (\mathbf{Set})$ which sends G to the set G.
4. The free functor $G : (\mathbf{Set}) \to (\mathbf{Grp})$ which sends the set S to the free group generated by the symbols in the set S.
5. The fundamental group functor $\pi_1 : (\mathbf{Top}^*) \to (\mathbf{Grp})$ which sends (X,x) to the group of loops start and end at x. The morphisms are the natural ones. For example $\pi_1(S^1, 1) = \mathbb{Z}$.

Let's see some actions

Theorem 1.9: Brouwer's Fixed Point Theorem

Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$ and $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Then if $f : D \to D$ is continuous then f has a fixed point.

Proof. Suppose not, and let $f : D \to D$ be continuous without fixed point. Then $f(x) \neq x$ for all $x \in D$, and thus the ray $\overline{f(x) \to x}$ must hit a unique point on S^1. Denote this point by $\phi(x)$, and observe $\phi : D \to S^1$ is continuous. In particular, note we have diagram

$$\begin{array}{ccc} S^1 & \xhookrightarrow{i} & D \\ & \searrow^{\mathrm{Id}_S} & \downarrow \phi \\ & & S^1 \end{array}$$

Now, by the above example, we see $\pi_1(S^1, 1) = \mathbb{Z}$, and by basic algebraic topology we see $\pi_1(D, 1) = \{1\}$ is trivial. Now by functoriality, we must have

$$\begin{array}{ccc} \pi_1(S^1,1) & \xhookrightarrow{\pi_1(i)} & \pi_1(D,1) \\ & \searrow^{\pi_1(\mathrm{Id}_S)} & \downarrow \pi_1(\phi) \\ & & \pi_1(S^1,1) \end{array}$$

This is the same as a diagram

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \{1\} \\ & \searrow^{\mathrm{Id}_{\mathbb{Z}}} & \downarrow \\ & & \mathbb{Z} \end{array}$$

There are no such arrow $\{1\} \to \mathbb{Z}$ make the above diagram commute, hence we are done.

Example 1.10

Let $S = R[x_1, \ldots, x_m]$ be the ring of polynomials over a commutative ring R. Let $I \subseteq S$ be an ideal of S. Then we can define the functor $V(I) : (R\text{-}\mathbf{Alg}) \to (\mathbf{Set})$ as follows: for each R-algebra $f : R \to T$, we let

$$V(I)(T) := \{(t_1, \ldots, t_m) \in T^m : f(t_1, \ldots, t_m) = 0 \forall f \in I\}$$

One can show this is a functor.

For example, if we let $R = \mathbb{R}$, $S = \mathbb{R}[x, y]$, and $I = \langle x^2 + y^2 - 1 \rangle$, then we see

$$V(I)(\mathbb{R}) = \{(x_0, y_0) \in \mathbb{R}^2 : x_0^2 + y_0^2 - 1 = 0\}$$

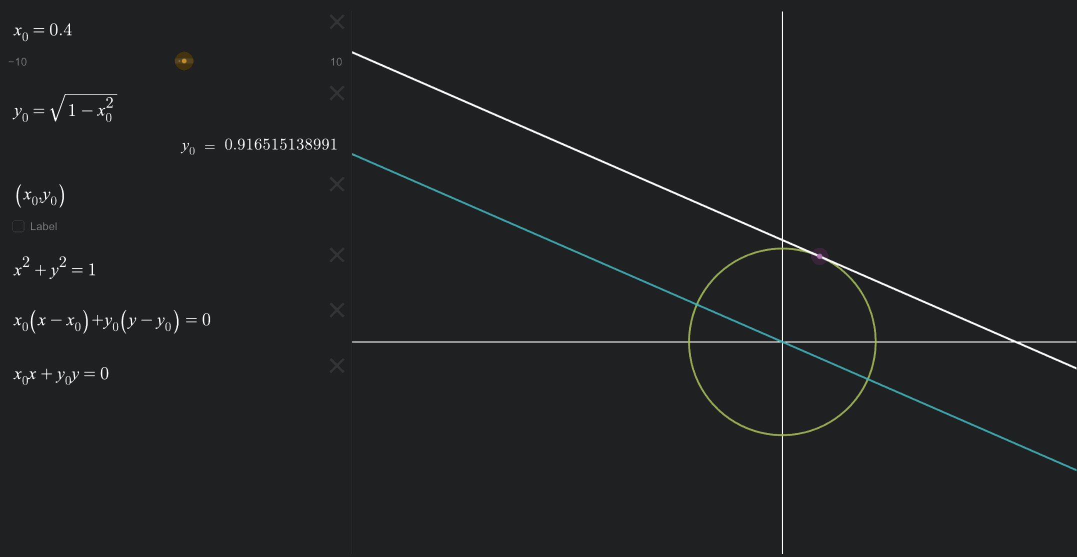
When we plot this set, its not hard to see the picture is given by the circle of radius 1, which we will denote by S^1. On the other hand, if we take the $\mathbb{R}$-algebra be $\mathbb{R}[t]/(t^2)$, which we will denote as $\mathbb{R}[\epsilon]$ where we use ϵ to denote the image of t in $\mathbb{R}[t]/(t^2)$. We see clearly that elements of $\mathbb{R}[\epsilon]$ can be written as $x_0 + x_1\epsilon$ for $x_0, x_1 \in \mathbb{R}$, such that $\epsilon^2 = 0$. Then we see $V(I)(\mathbb{R}[\epsilon])$ consists of all the tuples $(x_0 + x_1\epsilon, y_0 + y_1\epsilon)$, with $x_0, x_1, y_0, y_1 \in \mathbb{R}$, such that $(x_0 + x_1\epsilon)^2 + (y_0 + y_1\epsilon)^2 = 1$. Expand this expression out, we get

$$\begin{aligned}(x_0 + x_1\epsilon)^2 + (y_0 + y_1\epsilon)^2 &= x_0^2 + 2x_0x_1\epsilon + x_1^2\epsilon^2 + y_0^2 + 2y_0y_1\epsilon + y_1^2\epsilon^2 \\ &= (x_0^2 + y_0^2) + 2(x_0x_1 + y_0y_1)\epsilon\end{aligned}$$

For this expression to equal 1, we must have

$$\begin{cases} x_0^2 + y_0^2 = 1 \\ x_0x_1 + y_0y_1 = 0 \end{cases}$$

Now fix a point (x_0, y_0) on the circle S^1 of radius 1 in $\mathbb{R}^2$, we see all solutions to $x_0x_1 + y_0y_1 = 0$ lies on the unique line passing through origin and parallel to the tangent line of S^1 at (x_0, y_0): see the following



Example 1.11

1. Let $U : (\mathbf{Ring}) \longrightarrow (\mathbf{Grp})$ be $R \mapsto R^*$, which sends the ring R to the group of units of R. In this case, $f : R \to S$ then $U(f)$ is $f|_{R^*}$.
2. Let $F : (\mathbf{Ring}) \longrightarrow (\mathbf{Ab})$ be $R \mapsto R$ where we forget the ring structure on R

and only keep the additive structure. This is a functor.

3. Let $R : (\mathbf{Ab}) \to (\mathbf{Ring})$ be the free tensor $A \mapsto \bigoplus_{n\geq 0} A^{\otimes n}$. This is a functor.

Example 1.12

Let $(\mathbf{Dom})$ be the category of integral domains, and let $(\mathbf{Field}) \to (\mathbf{Dom})$ be the inclusion functor. Then, can we get an "inverse" functor $F : (\mathbf{Dom}) \to (\mathbf{Field})$ by define $F(R) = \mathrm{Frac}(R)$?

The answer is no, because for example, if $\mathbb{Z}[x] \to \mathbb{Z}$ is defined by $f(p(x)) = p(0)$. Then $F(f)(\frac{1}{x}) = \frac{f(1)}{f(x)} = \frac{1}{0}$, which makes no sense.

The fix is to consider $(\mathbf{Dom}_i)$, which is the category of integral domains, with morphisms being injective morphisms $R \to S$. Then the functor F is well-defined.

Definition 1.13

A functor $F : \mathcal{C} \to \mathcal{D}$ is ***full/faithful*** if $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(FX, FY)$ is surjective/injective. In the case F is full and faithful we say F is ***fully faithful***.

Let's see check whether some of the functors are full/faithful.

Example 1.14

Let $G : (\mathbf{Grp}) \to (\mathbf{Ab})$ be the abelianization.

Then this is not faithful. Indeed, take $A_5 \subseteq S_5$, and two morphisms $A_5 \to A_5$ be the identity and $h \mapsto xhx^{-1}$, conjugation by $x \neq e \in A_5$. Then we see $G(\mathrm{Id})$ and $G(h \mapsto xhx^{-1})$ are both the identity on $G(A_5)$, as $[A_5, A_5] = A_5$ because A_5 is simple and $[A_5, A_5]$ is normal and cannot be trivial (simple means the only normal subgroups of G is the trivial group and G itself).

This functor is also not full. Indeed, take

$$H_1 = \mathbb{Z}/2\mathbb{Z}, H_2 = \mathbb{Q}_8 = \{\pm i, \pm j, \pm k, \pm 1\}$$

Then $\mathrm{Hom}(H_1, H_2)$ consists of 4 elements and $\mathrm{Hom}(G(H_1), G(H_2))$ consists of 8 elements.

2 Natural Trans and Equivalence

Natural transformation is a way to compare functors.

Definition 2.1

Let $F, G : \mathcal{C} \to \mathcal{D}$, then we say $\theta : F \to G$ is a ***natural transformation*** if, for all $A \in \mathcal{C}$, we have an arrow $F(A) \xrightarrow{\theta_A} G(A)$, such that for all $f : A \to B$ we have diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle \theta_A} & & \downarrow{\scriptstyle \theta_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

We note that if $F : \mathcal{C} \to \mathcal{D}$, and $\theta = \mathrm{Id}_F$, then $\theta : F \to F$ is a natural trans because

$$\begin{array}{ccc} FA & \xrightarrow{F(f)} & FB \\ \downarrow{\scriptstyle \mathrm{Id}_{FA}} & & \downarrow{\scriptstyle \mathrm{Id}_{FB}} \\ FA & \xrightarrow{F(f)} & FB \end{array}$$

commutes.

We can compose two natural trans vertically, i.e. if we have

$$\begin{array}{ccc} F & & G \\ {\scriptstyle \eta}\downarrow & \Longrightarrow & \downarrow{\scriptstyle \epsilon} \\ G & & D \end{array}$$

then we can define $\epsilon \circ \eta : F \to D$ and this is again a natural transformation.

Example 2.2

Let $\mathcal{C}, \mathcal{D}$ be two categories, we can define $\mathcal{E} := \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ to be the category where objects are functors and morphisms are natural trans.

Example 2.3

Let $(\mathbf{Vec})$ be the moduli space of vector spaces, and $(\mathbf{Vec})(k)$ the category of k-vector spaces. Then we get

$$(\mathbf{Vec})(k) \to (\mathbf{Vec})(k)$$

$$V \mapsto V^*$$

This is clearly a functor. On the other hand, we can also do $V \mapsto V^{**}$, the double dual of V, which also defines a functor F. By linear algebra, we see there is a natural transformation between the double dual functor F and the identity functor.

Let's now restrict to $I : (\mathbf{FinVec})(k) \to (\mathbf{Vec})(k)$, where $(\mathbf{FinVec})$ is the moduli space of finite dimensional vector spaces, and I is the functor of inclusion that is fully faithful. In this subcategory, the natural transformation between Id and double dual F admits a inverse natural transformation. This is because when we deal with finite dimensional vector spaces, the dual and double dual will have the same dimension as V.

Example 2.4

Let G be a group, viewed as a category, and let $S : G \to (\mathbf{Sets})$ be a functor. It is not hard to see that such a functor is the same as a left G-set S, i.e. a set with a left G-action. Thus, suppose we have two functors $S, T : G \to (\mathbf{Sets})$, then a natural transformation $\theta : S \Rightarrow T$ is the same as a G-equivariant map $\theta : S \to T$. That is, $\theta(g \cdot s) = g \cdot \theta(s)$ for all $s \in S$ and $g \in G$.

Next we will talk about equivalence of categories and isomorphisms.

There should be a distinction between those two notions when we deal with categories. For example, consider a category $\mathcal{C}$ with objects being $\mathbb{Z}$ and we have an arrow $a \to b$ if and only if $a \equiv b \pmod 4$. On the other hand consider $\mathbb{Z}/4\mathbb{Z}$ as a category (as it is a groupoid). Those two categories should be equivalent, but they cannot be isomorphic as one has infinitely many objects and one has 4.

Definition 2.5

We say $\mathcal{C}$ and $\mathcal{D}$ are ***isomorphic*** if there are $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F = \mathrm{Id}_{\mathcal{C}}$ and $F \circ G = \mathrm{Id}_{\mathcal{D}}$.

Definition 2.6

We say $\mathcal{C}$ and $\mathcal{D}$ are ***equivalent*** if there are $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong \mathrm{Id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{Id}_{\mathcal{D}}$. In this case we say F is the ***quasi-inverse*** of G.

In other word, $\mathcal{C}$ and $\mathcal{D}$ are equivalent if we have $\eta : G \circ F \to \mathrm{Id}_{\mathcal{C}}$ and $\eta^{-1} : \mathrm{Id}_{\mathcal{C}} \to G \circ F$ and so on.

Exercise

Check the two categories we described above about $\mathbb{Z}/4\mathbb{Z}$ are equivalent. Conclude that

$$\text{PMATH945} \equiv \text{MATH145} \pmod{\mathbb{Z}/4\mathbb{Z}}$$

Definition 2.7

A functor $F : \mathcal{C} \to \mathcal{D}$ is ***essentially surjective*** if for every $D \in \mathcal{D}$, there is a $C \in \mathcal{C}$ such that $D \cong F(C)$.

Definition 2.8

A full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is a ***skeleton*** if for every object $X \in \mathcal{C}$, there is an isomorphism $X \xrightarrow{\sim} Y$ where $Y \in \mathcal{C}'$ and Y is unique. A category $\mathcal{C}$ is called a ***skeleton category*** if $\mathcal{C}$ is a skeleton of $\mathcal{C}$.

Lemma 2.9

If $\mathcal{C}$ is a category, then $\mathcal{C}$ admits a skeleton $\mathcal{C}'$. Moreover:

1. *The inclusion $\iota : \mathcal{C}' \to \mathcal{C}$ is an equivalence of category*
2. *If $\mathcal{C}$ and $\mathcal{D}$ are skeleton categories, then every fully faithful and essentially surjective $F : \mathcal{C} \to \mathcal{D}$ is an isomorphism of categories*

Proof. First we prove $\mathcal{C}'$ exists. By axiom of choice, we pick a representative in every isomorphism class in $\mathcal{C}$ (i.e. we consider $\mathcal{C}/\sim$ where $A \sim B$ iff $A \xrightarrow{\sim} B$). Let this category be $\mathcal{C}'$, which is a full subcategory of $\mathcal{C}$. Thus, for every $X \in \mathcal{C}$, we can use axiom of choice to select a particular isomorphism $\theta_X : X \xrightarrow{\sim} \kappa(X)$ with $\kappa(X) \in \mathcal{C}'$. WLOG we can pick $\theta_X = \mathrm{Id}_X$ if $X \in \mathcal{C}'$. Then there is a unique way to extend this map between objects $\kappa : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C}')$ to a functor so that $\theta : \mathrm{Id}_C \xrightarrow{\sim} \iota\kappa$. Indeed, set

$$\kappa(f) := \theta_Y \circ f \circ \theta_X^{-1} \in \mathrm{Hom}_{\mathcal{C}'}(\kappa(X), \kappa(Y)), \quad f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

On the other hand, we clearly have $\kappa\iota = \mathrm{Id}_{\mathcal{C}'}$, thus κ is quasi-inverse of ι, which shows ι is an equivalence of category.

On the other hand, suppose $F : \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective between skeleton categories. Then for any $Z \in \mathcal{D}$, we can find X so $Z \cong FX$, but then this means $Z = FX$ (as $\mathcal{D}$ is skeleton). Such an X is unique, because F is fully faithful and $FX \cong FX'$. Thus F is bijective on $\mathrm{Obj}(\mathcal{C})$ to $\mathrm{Obj}(\mathcal{D})$, and hence we can define its inverse G.

Theorem 2.10

Let $F : \mathcal{C} \to \mathcal{D}$, then F gives an equivalence of category if and only if F is essentially surjective and fully faithful.

Proof. Suppose F is an equivalence of categories. Take quasi-inverse $G : \mathcal{D} \to \mathcal{C}$ and $GF \underset{\sim}{\xrightarrow{\psi}} \mathrm{Id}_{\mathcal{C}}$ and $FG \underset{\sim}{\xrightarrow{\phi}} \mathrm{Id}_{\mathcal{D}}$. Then, for every $Z \in \mathcal{D}$ we can find $\phi_Z : F(G(Z)) \xrightarrow{\sim} Z$. This shows F is essentially surjective, and similarly G is essentially surjective. Now observe the following composition

$$\begin{array}{ccccccc} \mathrm{Hom}(X,Y) & \xrightarrow{F} & \mathrm{Hom}(FX,FY) & \xrightarrow{G} & \mathrm{Hom}(GF(X),GF(Y)) & \xrightarrow{\sim} & \mathrm{Hom}(X,Y) \\ f & \longmapsto & F(f) & \longmapsto & GF(f) & \longmapsto & \psi_Y GF(f) \psi_X^{-1} \end{array}$$

is in fact the identity, thus the first arrow F has left inverse, and the second arrow G has right inverse. Exchange the role of F and G, we see when $X, Y \in \mathrm{Obj}(\mathcal{C})$ lies in the image of G, then $\mathrm{Hom}(X,Y) \xrightarrow{F} \mathrm{Hom}(FX,FY)$ has right inverse. However, every object in $\mathcal{C}$ is isomorphic to something in the image of G, thus F is fully faithful as desired.

Now we prove the converse. By Lemma 2.9 we can take skeleton inclusions $\iota_1 : \mathcal{C}' \to \mathcal{C}$ and $\iota_2 : \mathcal{D}' \to \mathcal{D}$. Let κ_i be ι_i's quasi-inverse. Then $F' := \kappa_2 \circ F \circ \iota_1 : \mathcal{C}' \to \mathcal{D}'$ is fully faithful and essentially surjective. Thus we know F' is an isomorphism of categories (by Lemma 2.9, as $\mathcal{C}'$ and $\mathcal{D}'$ are both skeleton categories). Set $G := \iota_1 \circ F'^{-1} \circ \kappa_2$, then observe we have

$$GF = \iota_1 F'^{-1} \kappa_2 F \cong \iota_1 F'^{-1} \kappa_2 F \iota_1 \kappa_1 = \iota_1 \kappa_1 \cong \mathrm{Id}_{\mathcal{C}}$$

$$FG = F \iota_1 F'^{-1} \kappa_2 \cong \iota_2 \kappa_2 F \iota_1 F'^{-1} \kappa_2 = \iota_2 \kappa_2 \cong \mathrm{Id}_{\mathcal{D}}$$

where we used the fact $F' = \kappa_2 F \iota_1$.

Definition 2.11

Let $\mathcal{C}$ be a category, then we define the ***opposite category*** $\mathcal{C}^{\mathrm{opp}}$ be the category where $\mathrm{Obj}(\mathcal{C}) = \mathrm{Obj}(\mathcal{C}^{\mathrm{opp}})$ and the arrows are given by $A \to B$ in $\mathcal{C}$ if and only if we have an arrow $B \to A$ in $\mathcal{C}^{\mathrm{opp}}$, i.e. we revert all the arrows in $\mathcal{C}$.

Example 2.12

Let $\mathcal{C} = (\mathbf{FinVec})(\mathbb{C})$, then we have $\mathcal{C} \cong \mathcal{C}^{\mathrm{opp}}$ where we send $V \mapsto V^*$ and if $T : V \to W$ then T^* is given by $f \mapsto f \circ T$. On the other hand we have $\mathcal{C}^{\mathrm{opp}} \to \mathcal{C}$ by sending $V \mapsto V^*$ as well. This composition gives the double dual on $\mathcal{C}$ itself.

Example 2.13

Let $k = \overline{k}$ be ACF. The category of affine k-varieties is equivalent to the category of finitely generated reduced k-algebras. The two functors are given by $R \mapsto \max\mathrm{Spec}(R)$, and $X \mapsto \Gamma(X, \mathscr{O}_X)$.

Example 2.14

Similarly, we have Gelfand's theorem, which gives equivalence to the category of unital commutative C^*-algebras and the opposite category of the category of compact Hausdorff spaces. The two functors are similar, one is Spec and one is global sections.

Example 2.15

We also have Stone's theorem, which says the opposite category of category of Boolean algebras is equivalent to the compact totally disconnected Hausdorff spaces. To get from Boolean algebra to Stone spaces, we just take Spec, and on the other hand, we take X to the continuous functions from X to $\mathbb{F}_2$.

3 Adjoints

The adjoints are like nerds and jocks. I dont know what this means.

Definition 3.1

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, then we say F, G is an ***adjoint pair***, if we have natural isomorphism $\mathrm{Hom}_{\mathcal{D}}(FA, B) \xrightarrow{\alpha_{A,B}} \mathrm{Hom}_{\mathcal{C}}(A, GB)$ for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$.

Here natural means that, if we have $f : A \to A'$ and $g : B \to B'$ then

$$\begin{array}{ccc} \mathrm{Hom}(FA', B) & \xrightarrow{\alpha_{A',B}} & \mathrm{Hom}(A', GB) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(FA, B') & \xrightarrow{\alpha_{A,B'}} & \mathrm{Hom}(A, GB') \end{array}$$

In other words, for $x : FA' \to B$, we require the equality

$$G(g) \circ \alpha_{A',B}(x) \circ f = \alpha_{A,B'}(g \circ x \circ F(f))$$

We note natural can also be interprected as a natural transformation between $\mathrm{Hom}(F(-), -)$ and $\mathrm{Hom}(-, G(-))$.

Next we will talk about examples of adjunction.

Example 3.2

Let R be a commutative ring and let's look at the category of R-modules. Then recall the universal property of tensor product says that, given R-modules M, N, then $M \otimes N$ satisfies the condition that for all R-module P, whenever we have bilinear map $M \times N \to P$, then there is unique R-linear $M \otimes N \to P$.

Now take $A^B = \mathrm{Hom}_{(\mathbf{Sets})}(B, A)$, then for any $f : Y \times Z \to X$ in $X^{Y \times Z}$ we can map f to $\tilde{f} : Z \to X^Y$ by $\tilde{f}(z) = f(-, z)$. Now specialize to X, Y, Z be R-modules, then we get isomorphism

$$X^{Y \times Z} \cong (X^Y)^Z$$

In other word, we get

$$\mathrm{Hom}(Y \times Z, X) \cong \mathrm{Hom}_R(Z, \mathrm{Hom}_R(Y, X))$$

where on the left the arrows are R-bilinear. Now by universal property of $\otimes$, we see this is the same as saying

$$\mathrm{Hom}_R(Y \otimes_R Z, X) \cong \mathrm{Hom}_R(Z, \mathrm{Hom}_R(Y, X))$$

Recast this in terms of functors, let F, G be functors from R-modules to R-modules defined by:

1. $F(M) = Y \otimes_R M$
2. $G(M) = \mathrm{Hom}_R(Y, M)$

Then what we did above says

$$\mathrm{Hom}_R(F(Z),X) \cong \mathrm{Hom}_R(Z,G(X))$$

Let's check this is an adjoint pair. Suppose we have $\phi : Z \to Z'$ and $\psi : X \to X'$, then we want commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(Y\otimes Z',X) & \xrightarrow{\alpha_{Z',X}} & \mathrm{Hom}(Z',\mathrm{Hom}(Y,X)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(Y\otimes Z,X') & \xrightarrow{\alpha_{Z,X'}} & \mathrm{Hom}(Z,\mathrm{Hom}(Y,X')) \end{array}$$

Then we see for $f : Y \otimes Z' \to X$, we get

$$\alpha_{Z',X}(f) = \tilde{f} : Z' \to \mathrm{Hom}(Y,X)$$
$$\tilde{f}(z)(y) = f(y \otimes z)$$

Then, we see for any input $y \in Y$, we get

$$\begin{aligned} G(\psi)\circ\tilde{f}\circ\phi(z)(y) &= G(\psi)\circ\tilde{f}(\phi(z)) \\ &= G(\psi)(f(-\otimes\phi(z))) \\ &= \psi(f(y\otimes\psi(z))) \end{aligned}$$

On the other hand,

$$\begin{aligned} \psi\circ\tilde{f}\circ F(\phi)(z)(y) &= \psi\circ f\circ F(\phi)(y\otimes z) \\ &= \psi\circ f(y\otimes\phi(z)) \end{aligned}$$

The next example is field of fraction:

Example 3.3

Let $(\mathbf{Dom}_o)$ be the category of integral domains, with arrows being injections. Then let $F : (\mathbf{Dom}_o) \to (\mathbf{Fields})$ be the functor sends A to its field of fraction. On the other hand, let $G : (\mathbf{Fields}) \to (\mathbf{Dom}_o)$ be the forgetful functor. Then we see

$$\mathrm{Hom}(F(A),K) \cong \mathrm{Hom}(A,G(K)) = \mathrm{Hom}(A,E)$$

naturally, as one should check.

Example 3.4

Let $G : (\mathbf{Rings}) \to (\mathbf{Grp})$ be the functor defined by $G(R) = R^\times$, the unit group of R. On the other hand, we can define $F : (\mathbf{Grp}) \to (\mathbf{Rings})$ by

$$H \mapsto \mathbb{Z}[H]$$

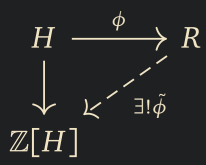
which is the group algebra of H over ring $\mathbb{Z}$. They forms an adjoint pair. Indeed,

$F(H) = \mathbb{Z}[H]$, then for $\phi : H_1 \to H_2$ we get

$$F(\phi) = \tilde{\phi} : \mathbb{Z}[H_1] \to \mathbb{Z}[H_2]$$
$$\sum_{h \in H_1} \alpha_h h \mapsto \sum_{h \in H_1} \alpha_h \phi(h)$$

Then we see $\mathrm{Hom}(F(H), R) \cong \mathrm{Hom}(H, G(R))$ naturally as one should check.

Its not hard to see all of those have an universal property at play. For example, for the unit group functor, we have



Example 3.5

Next, let's ask if $\pi_1 : (\mathbf{Top}^*) \to (\mathbf{Grp})$ has any left/right adjoint.

Is π_1 a left adjoint? If it were, we would get $G : (\mathbf{Grp}) \to (\mathbf{Top}^*)$ such that

$$\mathrm{Hom}_{(\mathbf{Grp})}(\pi_1(X, x_0), H) \cong \mathrm{Hom}_{(\mathbf{Top}^*)}((X, x_0), GH)$$

Now take

$$(S^1 \backslash \{-1\}, 1) \to (S^1, 1) \to (D, 1)$$

where D is the unit disk. Then we know their fundamental groups are given by

$$\{1\} \to \mathbb{Z} \to \{1\}$$

Now apply adjoint, we must have

$$\mathrm{Hom}(X, GH) \cong \mathrm{Hom}(\{1\}, H)$$

$$\mathrm{Hom}(Y, GH) \cong \mathrm{Hom}(\mathbb{Z}, H)$$

$$\mathrm{Hom}(Z, GH) \cong \mathrm{Hom}(\{1\}, H)$$

This is impossible as there is only one unique $X \to GH$, while we have many $Y \to GH$. Similarly we cannot make π_1 to be a right adjoint. Indeed, say F is left adjoint to π_1, then we get

$$\mathrm{Hom}(F\mathbb{Z}, X) \cong \mathrm{Hom}(\mathbb{Z}, \{1\})$$

$$\mathrm{Hom}(F\mathbb{Z}, Y) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$$

$$\mathrm{Hom}(F\mathbb{Z}, Z) \cong \mathrm{Hom}(\mathbb{Z}, \{-1\})$$

which is impossible.

In general, forgetful functor is adjoint to free functor.

Example 3.6

Let $G : (\textbf{Grp}) \to (\textbf{Set})$ be the forgetful functor, then it is adjoint to the functor which assigns a set S to the free group $\langle X \rangle$ generated by the set S. This also has a universal property: for all groups H and set map $h : X \to H$, we have

$$\begin{array}{ccc} X & \xhookrightarrow{x \mapsto x} & \langle X \rangle \\ \big\downarrow h & \swarrow \exists! \tilde{h} & \\ H & & \end{array}$$

There are many examples of free functors, which we will not go into details as they are very common. Another example would be $X \mapsto \text{span}(X)$, which sends a set X to the free vector space generated by X, so on and so on.

Example 3.7

Here is a slightly more complicated example. Let $G : (\textbf{Top}) \to (\textbf{Set})$ be the forgetful functor, and F be the functor sends a set S to the topological space S with discrete topology. Then observe

$$\text{Hom}(FS, X) \cong \text{Hom}(S, GX) = \text{Hom}(S, X)$$

as one should check.

The abelianization functor is adjoint to the forgetful functor. Indeed, this is because we have a universal property: for all group homomorphism $\phi : H \to A$ with A abelian, there exists a unique $\tilde{\phi} : H/[H,H] \to A$, which commutes with the projection map $H \to H/[H,H]$.

Example 3.8

Recall a Lie algebra is a vector space L equipped with Lie bracket. Then for any k-algebra R, we can define $[a, b] := ab - ba$ which makes R into a Lie algebra. Now, we have what's called the universal enveloping algebra, which sends a Lie algebra L over k to $U(L)$, with the universal property that, for all Lie algebra homomorphism $\phi : L \to R$ (where R is k-algebra), we have diagram

$$\begin{array}{ccc} L & \longrightarrow & U(L) \\ \big\downarrow \phi & \swarrow \exists! \tilde{\phi} & \\ R & & \end{array}$$

The construction of $U(L)$ is very strightforward, i.e. we define $U(L) = k\langle X \rangle / (xy - yx : x, y \in X)$, where X is any basis of L.

Example 3.9

Another example from analysis: let $F : (\textbf{MetricSp}) \to (\textbf{CMetricSp})$ be the completion functor which sends the metric space to its completion. Then this is left adjoint of the forgetful functor G.

Enough examples, now we do some theories.

Definition 3.10

Let (F, G, ϕ) be a adjoint pair, define the ***unit***

$$\eta : \mathrm{Id}_{\mathcal{C}} \to GF$$

by the image of Id_{FX} under $\phi : \mathrm{Hom}_{\mathcal{D}}(FX, FX) \to \mathrm{Hom}_{\mathcal{C}}(X, GFX)$, i.e. $\eta_X := \phi(\mathrm{Id}_{FX}) : X \to GFX$. Similarly we define the ***counit***

$$\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$$

by the image of Id_{GY} under $\phi^{-1} : \mathrm{Hom}_{\mathcal{C}}(GY, GY) \to \mathrm{Hom}_{\mathcal{D}}(Y, FGY)$, i.e. $\epsilon_Y := \phi^{-1}(\mathrm{Id}_{GY})$.

One can verify η and ϵ are natural transformations. This is mostly diagram chasing. Indeed, take $h : X' \to X$ in $\mathcal{C}$, then by naturality of ϕ, we get commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Id}_{FX} & & \mathrm{Hom}(FX, FX) & \xrightarrow{\phi} & \mathrm{Hom}(X, GFX) & & \eta_X \\
\downarrow & & \downarrow{\scriptstyle (Fh)^*} & & \downarrow{\scriptstyle h^*} & & \downarrow \\
Fh & & \mathrm{Hom}(FX', FX) & \xrightarrow{\phi} & \mathrm{Hom}(X', GFX) & & \phi(Fh) \\
\uparrow & & \uparrow{\scriptstyle (Fh)_*} & & \uparrow{\scriptstyle (GFh)_*} & & \uparrow \\
\mathrm{Id}_{FX'} & & \mathrm{Hom}(FX', FX') & \xrightarrow[\phi]{} & \mathrm{Hom}(X', GFX') & & \eta_{X'}
\end{array}$$

where $(Fh)^*$ means it sends $f : FX \to FX$ to $f \circ Fh$ and $(Fh)_*$ means sends $f : FX' \to FX'$ to $(Fh) \circ f$ and so on. By chasing the diagram we conclude the following diagram

$$\begin{array}{ccc}
X' & \xrightarrow{\eta_{X'}} & GFX' \\
{\scriptstyle h}\downarrow & \searrow^{\phi(Fh)} & \downarrow{\scriptstyle GFh} \\
X & \xrightarrow[\eta_X]{} & GFX
\end{array}$$

is commutative. This shows η is a natural trans.

We also observe the unit and counit determines ϕ in the following way:

$$\begin{aligned} \phi(f) &= Gf \circ \eta : X \to GY, \quad \forall f : FX \to Y \\ \phi^{-1}(g) &= \epsilon_Y \circ Fg : FX \to Y, \quad \forall g : X \to GY \end{aligned} \qquad \text{(Eq. 3.1)}$$

We can prove this using a diagram: for $\phi(f)$, observe

$$\begin{array}{ccccccc}
\mathrm{Id}_{FX} & & \mathrm{Hom}(FX, FX) & \xrightarrow{\phi} & \mathrm{Hom}(X, GFX) & & \eta_X \\
\downarrow & & {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (Gf)_*} & & \downarrow \\
f & & \mathrm{Hom}(FX, Y) & \xrightarrow[\phi]{} & \mathrm{Hom}(X, GY) & & \phi(f) = Gf \circ \eta_X
\end{array}$$

The case for $\phi^{-1}(g)$ is similar.

Lemma 3.11

For adjoint pair (F, G, ϕ), unit η and counit ϵ, we have equations

$$\begin{aligned} \mathrm{Id}_G &= \left(G \xrightarrow{\eta \circ G} (GF)G = G(FG) \xrightarrow{G \circ \epsilon} G\right) \\ \mathrm{Id}_F &= \left(F \xrightarrow{F \circ \eta} F(GF) = (FG)F \xrightarrow{\epsilon F} F\right) \end{aligned} \tag{Eq. 3.2}$$

Proof. Let $Y \in \mathcal{D}$, by definition of $\epsilon_Y : FGY \to Y$ and Equation Eq. 3.1, we see

$$\mathrm{Id}_{GY} = \phi(\epsilon_Y) = G(\epsilon_Y) \circ \eta_{GY} : GY \to GY$$

This is the first claim we want to prove. Using $\mathrm{Id}_{FX} = \phi^{-1}(\eta_X)$ and Equation Eq. 3.1 we get the second claim.

Proposition 3.12

For a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, the following maps are inverse to each other:

$$\{\varphi : (F, G, \varphi) \text{ is an adjoint pair }\} \rightleftharpoons \{(\eta, \varepsilon) : \text{ satisfies Eq.3.2}\}$$
$$\varphi \longrightarrow \left(\eta_X := \varphi\left(\mathrm{id}_{FX}\right), \varepsilon_Y := \varphi^{-1}\left(\mathrm{id}_{GY}\right)\right)$$
$$\varphi(f) := Gf \circ \eta_X \longleftarrow (\eta, \varepsilon)$$

Proof. Given units and counits (η, ϵ). Define $\phi(f) = Gf \circ \eta_X$ and $\psi(g) = \epsilon_Y \circ Fg$, where $f : FX \to Y$ and $g : X \to GY$. By naturality of η and ϵ, we see ϕ and ψ gives a natural transformation between functors $\mathrm{Hom}(F(-), -)$ and $\mathrm{Hom}(-, G(-))$. We claim $\psi \circ \phi = \mathrm{Id}$. Indeed, the left map sends f to $\epsilon_Y \circ FGf \circ F\eta_X$. By naturality of ϵ, we see the diagram

$$\begin{array}{ccccc} FX & \xrightarrow{F\eta_X} & FGFX & \xrightarrow{FGf} & FGY \\ & & \downarrow{\scriptstyle \epsilon_{FX}} & & \downarrow{\scriptstyle \epsilon_Y} \\ & & FX & \xrightarrow[f]{} & Y \end{array}$$

commutes. Thus $\psi\phi(f) = f \circ \epsilon_{FX} \circ F\eta_X$. By Equation Eq. 3.2 we see this is f as desired, i.e. $\psi\phi = \mathrm{Id}$. Similarly we see $\phi\psi = \mathrm{Id}$. This together with the above discussion we conclude the proof.

Theorem 3.13

Suppose $F : \mathcal{C} \to \mathcal{D}$ be a functor. If $G, G' : \mathcal{D} \to \mathcal{C}$ be two right adjoints of F, then

$G \cong G'$.

Proof. By definition of adjoint, we have

$$\mathrm{Hom}(A, GB) \xrightarrow{\alpha_{A,B}} \mathrm{Hom}(FA, B) \xleftarrow{\alpha'_{A,B}} \mathrm{Hom}(A, G'B)$$

We want to produce a map $\eta : G \to G'$ so for all $D \in \mathcal{D}$ we get $\eta_D : GD \to G'D$.

But take $A = GD$ and $B = D$, we see by definition of adjoint, we get

$$\begin{aligned} \mathrm{Hom}(GD, GD) &\xrightarrow{\sim} \mathrm{Hom}(GD, GD') \\ \mathrm{Id}_{GD} &\mapsto \eta_D \end{aligned}$$

which is obtained by compose α and $(\alpha')^{-1}$.

It remains to show η is natural. Thus let $f : D \to D'$, then we want to check if the following diagram

$$\begin{array}{ccc} GD & \xrightarrow{Gf} & GD' \\ \downarrow \eta & & \downarrow \eta \\ G'D & \xrightarrow{G'f} & G'D' \end{array}$$

is commutative. But by naturality of adjoint, we get commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(A', GB) & \longrightarrow & \mathrm{Hom}(FA', B) & \longleftarrow & \mathrm{Hom}(A', G'B) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}(A, GB') & \longrightarrow & \mathrm{Hom}(FA, B') & \longleftarrow & \mathrm{Hom}(A, G'B') \end{array}$$

for all $A \to A'$ and $B \to B'$. Now set $A = A' = GD$, $\mathrm{Id} : A \to A'$, $B = D$ and $B' = D'$ with $f : B \to B'$. Then we get a diagram

$$\begin{array}{ccc} \mathrm{Hom}(GD, GD) & \longrightarrow & \mathrm{Hom}(GD, G'D) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(GD, GD') & \xrightarrow[\beta]{} & \mathrm{Hom}(GD, G'D') \end{array}$$

where $\mathrm{Id}_{GD} \in \mathrm{Hom}(GD, GD)$ will be mapped to $G'f \circ \eta_D$ if we go right then down. But if we go down then right we dont really get what we wanted. To complete the proof, we use naturality again, with $B = B' = GD'$, $\mathrm{Id} : B \to B'$, $A = GD$ and $A' = GD'$ with $Gf : A \to A'$. This gives

$$\begin{array}{ccc} \mathrm{Hom}(GD', GD') & \longrightarrow & \mathrm{Hom}(GD', G'D') \\ \downarrow & & \downarrow \\ \mathrm{Hom}(GD, GD') & \xrightarrow[\beta]{} & \mathrm{Hom}(GD, G'D') \end{array}$$

where if we go down then right, we get the same arrow as if we go down then right in the diagram above this diagram. In other word, if we go right then down, we would get an arrow equal to $G'f \circ \eta_D$. But if we chase the diagram, this is in fact $\eta_{D'} \circ Gf$. This proves the naturality of η, as desired.

It remains to show we can find $\epsilon : G' \to G$ so $\epsilon \circ \eta = \mathrm{Id}_{G'}$ and $\eta \circ \epsilon = \mathrm{Id}_G$. This is also just diagram chasing.

Now we do another example of adjoints.

Example 3.14

Let X be topological space, then we get category $\mathrm{Op}(X)$, with objects being open sets and arrows being inclusion relation, i.e. $U \to V$ if and only if $V \subseteq U$. Let $\mathcal{D}$ be a concrete category, such as $(\mathbf{Set})$, $(\mathbf{Grp})$, $(\mathbf{Ring})$, $(R\text{-}\mathbf{Mod})$ and so on.

Then a presheaf valued in $\mathcal{D}$ is a functor $\mathscr{F} : \mathrm{Op}(X) \to \mathcal{D}$. We say $\mathscr{F}$ is a sheaf if for all U and open cover $U = \bigcup U_i$, we get short exact sequence

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

where p_1 sends $(s_i)_i$ to $(s_i|_{U_i \cap U_j})_{i,j}$ and p_2 sends $(s_i)_i$ to $(s_j|_{U_i \cap U_j})_{i,j}$. Its not hard to see $\mathscr{F}(\emptyset)$ must be the terminal object in the category its valued in.

Since presheaves are just functors from $\mathrm{Op}(X) \to \mathcal{D}$, we see morphisms of presheaves are just natural transformations $\eta : \mathscr{F} \to \mathscr{G}$. In particular we get a functor $(\mathbf{Sh}) \to (\mathbf{PreSh})$, which is just the inclusion of sheaves into presheaves (i.e. it is the forgetful functor). This functor has an adjoint, which is called sheafification.

To define sheafification, one way to do it is using the explicit notion of stalks. We define this as, for $x \in X$,

$$\mathscr{F}_x := \varinjlim_{x \in V} \mathscr{F}(V)$$

where the colimit runs over all open sets V containing x. This can also be defined as equivalence class of $[(f, U)]$ with $f \in \mathscr{F}(U)$ (we omit the equivalence relation).

Then, for a presheaf $\mathscr{F}$, we define its sheafification $\mathscr{F}^{\mathrm{sh}}$ by

$$\mathscr{F}^{\mathrm{sh}}(U) := \left\{ (s_x) \in \prod_{x \in U} \mathscr{F}_x : (s_x) \text{ compatible} \right\}$$

Here compatible means the following: for all $x \in U$, we can find open nbhd V_x of x and $t \in \mathscr{F}(V_x)$, such that $s_z = [(t, V_x)]$ for all $z \in V_x$.

Proposition 3.15

Let $\mathscr{F}$ be a presheaf, then $\mathscr{F}^{\mathrm{sh}}$ is a sheaf.

Proof. Standard (and annoying) routine.

Theorem 3.16

The forgetful functor $(\textbf{\textit{Sh}}) \to (\textbf{\textit{PreSh}})$ *is right adjoint of* $\mathrm{sh} : (\textbf{\textit{PreSh}}) \to (\textbf{\textit{Sh}})$.

4 Yoneda Lemma

Let $\mathcal{C}$ be locally small category, and consider $\mathrm{Hom}(-,-) : \mathcal{C}^{\mathrm{opp}} \times \mathcal{C} \to (\textbf{Sets})$. In particular by fixing $A \in \mathcal{C}$ we get $h_A := \mathrm{Hom}(A, -)$, which is a functor from $\mathcal{C}^{\mathrm{opp}}$ to $(\textbf{Sets})$. Similarly we can do $h^A = \mathrm{Hom}(-, A)$.

Theorem 4.1: Yoneda

Let $\mathcal{F}$ *be the full subcategory of* $\mathrm{Fun}(\mathcal{C}, (\textbf{Sets}))$ *with objects* h^A. *Then we have isomorphism of categories* $\mathcal{C} \cong \mathcal{F}^{\mathrm{opp}}$. *Similarly we get* $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{opp}}, (\textbf{Sets}))$.

We note in general its not enough to just know the size of the hom set, as we also need naturality. On the ot her hand, it is, actually enough when working with finite abelian groups, i.e. we know $A = \prod_i \mathbb{Z}/p_i^{k_i}\mathbb{Z}$ for some $k_i \geq 1$. But then $\mathrm{Hom}(A, \mathbb{Z}/p\mathbb{Z}) = p^r$ where $r = \#\{j : p_j \geq p\}$ and so on.

Definition 4.2

We say a functor $F : \mathcal{C}^{\mathrm{opp}} \to (\textbf{Set})$ is ***representable***, if there exists $X \in \mathcal{C}$ and isomorphism $\phi : h_X \xrightarrow{\sim} F$.

Example 4.3

Take $\mathcal{C} = \mathrm{Op}(X)$ be the category of open sets on topological space X. Then we see it sends an open set U to a presheaf on X, namely $\mathscr{F}(V) = \emptyset$ if $U \nsubseteq V$ and otherwise its just the restriction of sections.

Theorem 4.4

A small category $\mathcal{C}$ *is isomorphic to a concrete category.*

Proof. Let $Y : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}, (\textbf{Sets}))^{\mathrm{opp}}$ be the functor $A \mapsto h_A$. Then we see since $\mathcal{C}$ is small, $h_A \cong \coprod_{B \in C} \mathrm{Hom}(A, B)$. Thus we can form a new category $\mathcal{C}'$ with objects being $\coprod_{B \in C} \mathrm{Hom}(A, B)$ as A varying. One checks if we have $\phi : A \to A'$ then we get induced map in $\mathcal{C}'$. But then its strightforward to see we get an equivalence of categories $\mathcal{C}$ and $\mathcal{C}'$.

5 Limits and Colimits

Definition 5.1

An object $I \in \mathcal{C}$ is ***initial*** if there exists unique $I \to A$ for all $A \in \mathcal{C}$. An object is ***terminal*** if its initial in $\mathcal{C}^{\text{opp}}$.

We note if I is initial then its unique up to unique isomorphism.

Example 5.2

The following examples are standard

	initial	terminal
(**Sets**)	$\emptyset$	$\{*\}$
(**Rings**)	$\mathbb{Z}$	(0)
(**AbGrp**)	(0)	(0)

On the other hand, in (**Fields**), we have no initial or terminal objects. When we take the category of k-algebras, then initial is k and terminal is (0).

We can define initial and terminal objects by adjoint pairs. Let $\bullet$ be the category with 1 object $\bullet$ and 1 morphism $\text{Id}_\bullet$. Then we get unique $G : \mathcal{C} \to \bullet$ by sending A to $\bullet$. Then, the functor which sends $\bullet$ to I is a left adjoint to G. On the other hand, the functor sends $\bullet$ to T is a right adjoint to G.

In our definition of limits and colimits, we will use cone and cocones.

Definition 5.3

A ***diagram*** is a functor $\mathcal{B} \to \mathcal{C}$.

Typically, $\mathcal{B}$ will be a small category. In this case, we can think of $\mathcal{B}$ as a bunch of dots together with arrows between each other, as we can make $\mathcal{B}$ into a concrete category.

Definition 5.4

Let $F : \mathcal{B} \to \mathcal{C}$ be a diagram. A ***cocone*** to F is an object N of $\mathcal{C}$ and maps $\phi_B : FB \to N$ for all $B \in \mathcal{B}$, such that the diagram commutes with N.

To be explicit, when we say the diagram commutes with N, we mean that whenever we have $FB \to FB'$, then the triangle

$$\begin{array}{ccc} FB & \longrightarrow & FB' \\ \downarrow & \swarrow & \\ N & & \end{array}$$

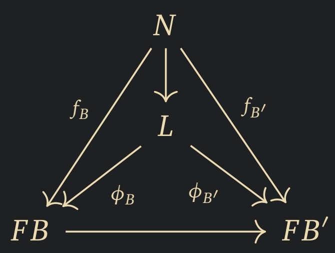
commutes.

Definition 5.5

Let $F : \mathcal{B} \to \mathcal{C}$ be a diagram, then a ***cone*** to F is a cocone to $\mathcal{B} \to \mathcal{C}^{\text{opp}}$, i.e. we want object N and morphisms $\phi_B : N \to FB$ such that it commutes with the diagram.

Definition 5.6

A ***limit*** of a diagram F is a cone (L, ϕ_B) such that, if (N, f_B) is another cone, then there exists unique $\theta : N \to L$ such that the following diagram commutes



for all $B \to B'$. In this case we write L as

$$\varprojlim F := L$$

The definition of colimit is where we invert all the arrows above (i.e. we work with cocones). In this case we write $\varinjlim F$.

Remark 5.7

Note given $F : \mathcal{B} \to \mathcal{C}$ a diagram, then we get a category of cones (resp. cocones). If we do this, then the limits are just terminal objects in the category of cones, and colimits are just initial objects in the category of cocones.

Note if we take $\mathcal{B}$ be the empty diagram, then we just get the initial and terminal objects of $\mathcal{C}$, if they exists.

Example 5.8

Let $\mathcal{B}$ be the category with morphisms consists of only identities. Then we see

$$\varprojlim F = \prod_{B \in \mathcal{B}} FB$$

$$\varinjlim F = \coprod_{B \in \mathcal{B}} FB$$

If we take $\mathcal{C}$ to be the category of sets, then we just get $\varprojlim F$ is Cartesian product of sets and $\varinjlim F$ the disjoint union of sets. When we take $\mathcal{C} = (\mathbf{Top})$, then the product is literally $\prod X_i$ with product topology. In the category of groups, product is just free product and coproduct is direct sum. In category of commutative rings, product is Cartesian product and coproduct is tensor product.

In rings, $\mathbb{Z}[x]\coprod\mathbb{Z}[y]=\mathbb{Z}\langle x,y\rangle$ and in commutative rings $\mathbb{Z}[x]\coprod\mathbb{Z}[y]=\mathbb{Z}[x,y]$.

Lemma 5.9

Let $I=\coprod_{j\in J}I_j$ as a partition of sets. If all products indexed by $\mathcal{I}_j$ exists for all $j\in J$, and all products indexed by J exists, then all products indexed by I exists in $\mathcal{C}$. In particular we have unique isomorphism α and commutative diagram

$$\begin{array}{ccc} \prod_I X_i & \xrightarrow{\alpha} & \prod_{j\in J}\left(\prod_{i\in I_j}X_i\right) \\ & \searrow^{p_k} & \downarrow{\scriptstyle p_k p_j} \\ & & X_k \end{array}$$

Similar results holds for coproducts.

Example 5.10

Let $\mathcal{I}$ be a small category with diagram $F:\mathcal{I}\to(\mathbf{Sets})$. Then

$$\varprojlim F := \left\{(x_i)_{i\in\mathrm{Obj}(\mathcal{I})}\in\prod_{i\in\mathcal{I}}F(i):\forall\sigma\in\mathrm{Hom}_{\mathcal{I}}(i,j),F(\sigma)(x_j)=x_i\right\}$$

This is, in fact, an equalizer diagram, which we will define later. Clearly we have natural projection $p_j:\varprojlim F\to F(j)$, defined by $p_j((x_i)_i)=x_j$, where j range over $\mathcal{I}$. Next we consider $G:\mathcal{I}\to(\mathbf{Set})$ and we compute $\varinjlim G$. This is given by

$$\varinjlim G := \left(\coprod_{i\in\mathcal{I}}G(i)\right)/\sim$$

where the equivalence relation is generated by

$$x\sim G(\sigma)(x),\quad \forall\sigma:i\to j,x\in G(i)$$

Those two examples above are nice, as we get a complete description of what limits and colimits look like when our category is at least sets. However, the $\sim$ relation here is very inexplicit. We will resolve this problem when we consider filtered limits/colimits.

Example 5.11

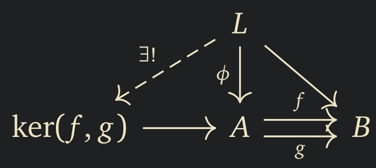
Let X be a topological space, $\mathscr{F}$ a sheaf on X. Then we see the stalk is an colimit.

Next, we talk about equalizer/coequalizer, and pushout/pullback.

Definition 5.12

Consider the diagram $\mathcal{I}$ defined by $\bullet \underset{b}{\overset{a}{\rightrightarrows}} \bullet$. Then for $F : \mathcal{I} \to \mathcal{C}$ and let $f = F(a)$ and $g = F(b)$. Then we define the ***coequalizer***, denoted by $\operatorname{coker}(f,g)$ as $\varinjlim F$. Similarly we define the ***equalizer***, denoted by $\ker(f,g)$, as $\varprojlim F$.

Let's explain what this is in more details. A functor $F : \mathcal{I} \to \mathcal{C}$ is the same as two arrows $f : A \to B$ and $g : A \to B$. Then, the universal property of $\ker(f,g)$ is given by the following diagram



That is, the two compositions

$$\left(\ker(f,g) \to A \xrightarrow{f} B\right) = \left(\ker(f,g) \to A \xrightarrow{g}\right)$$

are equal, and whenever we have $L \to B$ and $L \xrightarrow{\phi} A$ that makes the triangle commutes, the arrow $L \to B$ actually factor through $\ker(f,g)$. The case for $\operatorname{coker}(f,g)$ is just flip all the arrows, and hence left as an exercise.

Example 5.13

Let $\mathcal{C} = (\mathbf{Set})$ be the category of sets and $f,g : X \to Y$. Then

$$\ker(f,g) = \{x \in X : f(x) = g(x)\}$$

$$\operatorname{coker}(f,g) = Y/\sim$$

where the equivalence relation $\sim$ is generated by $f(x) \sim g(x)$.

Lemma 5.14

Let $f,g : X \to Y$. If $\ker(f,g)$ exists, then $\ker(f,g) \to X$ is a monomorphism. If $\operatorname{coker}(f,g)$ exists, then $Y \to \operatorname{coker}(f,g)$ is a epimorphism.

Here recall we say $f : X \to Y$ is monomorphism if $f^* : \operatorname{Hom}(Z,X) \to \operatorname{Hom}(Z,Y)$ is injective for all objects Z, and f is epimorphism if $f_* : \operatorname{Hom}(Y,Z) \to \operatorname{Hom}(X,Z)$ is injective for all objects Z.

Definition 5.15

Consider the diagram $\mathcal{I}$ defined by $\bullet \longleftarrow \bullet \longrightarrow \bullet$. Then we define:

1. the ***pullback*** (or ***fibered product***) as $\varprojlim F$ where $F : \mathcal{I}^{\mathrm{opp}} \to \mathcal{C}$ is given by $X \to Z \leftarrow Y$, and we denote this by $X \times_Z Y$
2. the ***pushout*** (or ***fibered coproduct***) as $\varinjlim F$ where $F : \mathcal{I} \to \mathcal{C}$ is given by $X \leftarrow Z \to Y$, and we denote this by $X \coprod_Z Y$

As usual, we record the universal property of fibered product (and pushout is just flip all the arrows): the pullback satisfies

$$\begin{tikzcd} L \arrow[rd, dashed, "\exists!"] \arrow[rrd, bend left] \arrow[rdd, bend right] & & \\ & X \times_Z Y \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d] \\ & Y \arrow[r] & Z \end{tikzcd}$$

We often denote the pullback diagram by

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

and for pushout we replace $\square$ by $\boxplus$

Example 5.16

Consider $\mathbb{A}^2_{\mathbb{C}}$ and the group $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$. This has some subgroups, such as L, which consists of linear automorphisms, and T, which consists of the automorphisms $(x,y) \mapsto (x+p(y),y)$. Let $A = L \cap T$, then a theorem says $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ is the pushout in $(\mathbf{Grp})$ of $L \hookleftarrow A \to T$.

Example 5.17

Consider the Seifert-Van Kampen theorem. Say we have pointed topological spaces X, Y with a common point x. This is the same as saying we have $X \cap Y = A$ and we have inclusion $(X,x) \leftarrow (A,x) \to (Y,x)$. In particular if we take colimit and then fundamental group, then Seifert-Van Kampen says this is the same as the colimit of fundamental group. That is, we have

$$\pi_1(\varinjlim(X \leftarrow A \to Y)) \cong \varinjlim(\pi_1(X \leftarrow A \to Y)) = \pi_1(X) *_{\pi_1(A)} \pi_1(Y)$$

Right, so we talked about limits and colimits, but in general we do not have a very nice description of them. Thus we consider filtered limits instead.

Let Λ be a directed set with $\leq$, i.e. we have $x \leq x, x \leq y \leq z$ and upper bound of x, y. Then let $\mathcal{C}$ be a category, then consider the category consists of $\{C_i\}_{i \in \Lambda}$ and morphisms being $(f_{ij} : C_i \to C_j)_{i \geq j}$, where $f_{ij} \circ f_{jk} = f_{ik}$ and $f_{ii} = \mathrm{Id}$. Then, the limit $\varprojlim C_i$, if it exists, is called the projective limit of the projective system.

Example 5.18

Consider the projections $\pi_n : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$ and the projective system

$$\ldots \to \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z} \to \ldots \to \mathbb{Z}/p\mathbb{Z}$$

Then the projective limit of the system on $\mathbb{N}$ formed by $\leq$ is $\mathbb{Z}_p$, the p-adic integers.

How can we describe this in another way? Well, we know

$$\mathbb{Z}_p \subseteq \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$$

where $(x_1, x_2, \ldots) \in \mathbb{Z}_p$ if and only if $\pi_2(x_2) = x_1$, $\pi_3(x_3) = x_2$, $\pi_4(x_4) = x_3$ and so on.

Example 5.19

The next example we consider is the profinite groups. The setup is again a directed set Λ and $\{G_i\}_{i\in\Lambda}$ with G_i finite groups. We work with the category $\mathcal{C}$, compact Hausdorff topological groups. Assume we have a projective system, then $\hat{G} = \varprojlim G_i$ is called a profinite group. Explicitly, $\hat{G} = \{(g_i)_{i\in\Lambda} : \forall i \geq j, f_{ij}(g_i) = g_j\}$. Since G_i is finite we can give this the discrete topology, and its going to be compact as its fintie. Thus the projective limit $\hat{G}$ is compact Hausdorff totally disconnected group.

Theorem 5.20: RAPL

Let $F : \mathcal{C} \to \mathcal{D}$ be the left adjoint to G. If $\varprojlim D_i$ exists in $\mathcal{D}$, then $\varprojlim GD_i$ exists in $\mathcal{C}$ and $\varprojlim GD_i \cong G(\varprojlim D_i)$.

To prove this, we need to develop some theory.

First, recall that by Yoneda we can embed any $\mathcal{C}$ into the category of functors. We now denote those by

$$\mathcal{C}^\wedge := \text{Fun}(\mathcal{C}^{\text{opp}}, (\textbf{Sets}))$$

$$\mathcal{C}^\vee = \text{Fun}(\mathcal{C}^{\text{opp}}, (\textbf{Sets})^{\text{opp}}) = \text{Fun}(\mathcal{C}, (\textbf{Sets}))^{\text{opp}}$$

Then Yoneda lemma says that, for $S \in \mathcal{C}$ and $A \in \mathcal{C}^\wedge$, the map

$$\text{Hom}_{\mathcal{C}^\wedge}(h_S, A) \to A(S)$$

$$\left(\text{Hom}_{\mathcal{C}}(\cdot, S) \xrightarrow{\phi} A(\cdot)\right) \mapsto \phi_S(\text{Id}_S)$$

is bijective, and it gives an isomorphism of functors $\text{Hom}_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(\cdot), \cdot) \xrightarrow{\sim} \text{ev}^\wedge$, where $\text{ev}^\wedge : (\mathcal{C}^{\text{opp}}) \times \mathcal{C}^\wedge \to (\textbf{Sets})$ is given by $(S, A) \mapsto A(S)$, and $h_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^\wedge$ sends S to $h^S = \text{Hom}(\cdot, S)$.

Now here comes one nice application of Yoneda lemma: given category $\mathcal{C}$, even a limit $\varprojlim F$ does not exists in $\mathcal{C}$, once we embed $\mathcal{C}$ into $\mathcal{C}^\wedge$ or $\mathcal{C}^\vee$, it always exists (to be exact, all small limits exists). This is the following proposition:

Proposition 5.21

Let $\mathcal{I}$ be a small category, then the limit of the diagram $F : \mathcal{I} \to \mathcal{C}^\wedge$ is given by the object "$\varinjlim$"$F \in \text{Obj}(\mathcal{C}^\wedge)$

$$\text{“}\varinjlim\text{”}F : S \mapsto \varinjlim(F(S))$$

$$\left(T \xrightarrow{f} S\right) \mapsto \left(\varinjlim F(S) \xrightarrow{\varinjlim F(f)} \varinjlim F(T)\right)$$

together with natural transformation $F(j)(\cdot) \to \varinjlim F(\cdot)$, *where* $j \in \mathrm{Obj}(\mathcal{I})$ *and* $\varinjlim F(f)$ *is the natural one. Similarly the limit of* $G : \mathcal{I}^{\mathrm{opp}} \to \mathcal{C}^{\wedge}$ *is given by* "$\varprojlim$"$G : S \mapsto \varprojlim(G(S))$ *and so on.*

Proof. The trick is point by point reduce to the case of sets. We only explain $F : \mathcal{I} \to \mathcal{C}^{\wedge}$ and leave $G : \mathcal{I}^{\mathrm{opp}} \to \mathcal{C}^{\vee}$ for the readers. Consider the universal property of colimits, i.e. we get

$$\begin{array}{ccccc} F(i) & & \xrightarrow{F(\phi)} & & F(j) \\ & \searrow^{\iota_i} & & \swarrow^{\iota_j} & \\ & & \varinjlim F & & \\ & \searrow & \downarrow{\scriptstyle \exists!} & \swarrow & \\ & & L & & \end{array}$$

where $L \in \mathcal{C}^{\wedge}$ and a bunch of arrows $F(i) \to L$. For every object $S \in \mathcal{C}$, since $\varinjlim$ exists in $(\textbf{Sets})$, we know there exists unique $\phi(S)$ such that the diagram

$$\begin{array}{ccc} F(i)(S) & \longrightarrow & \varinjlim F(S) \\ & \searrow & \downarrow{\scriptstyle \phi(S)} \\ & & L(S) \end{array}$$

is commutative for all $i \in \mathcal{I}$. Now it remains to prove those $\phi(\cdot)$ can be glued into "$\varinjlim$"$F \xrightarrow{\phi} L$. Let $f : T \to S$ be any arrow in $\mathcal{C}$, let's now treat $L(S)$ as the constant functor $\Delta(L(S)) : \mathcal{I} \to (\textbf{Sets})$. Its easy to see $\varinjlim \Delta(L(S)) = L(S)$. A similar claim holds for $L(T)$. Thus, we see for every $i \in \mathcal{I}$, the following diagram

$$\begin{array}{ccc} F(i)(S) & \xrightarrow{F(i)(f)} & F(i)(T) \\ \downarrow & & \downarrow \\ L(S) & \longrightarrow & L(T) \end{array}$$

commutes. In particular, by naturality of colimits and the commutative of above diagram, we conclude

$$\begin{array}{ccc} \varinjlim F(S) & \xrightarrow{\varinjlim F(f)} & \varinjlim F(T) \\ \downarrow{\scriptstyle \phi(S)} & & \downarrow{\scriptstyle \phi(S)} \\ L(S) & \longrightarrow & L(T) \end{array}$$

is commutative. But this is the same as saying ϕ is functorial, and we are done.

In particular, by the above result, we can translate the existence problem of limits and colimits into representability problem of functors, and this is what the following proposition says. Before we start, let $h_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^\wedge$ be the functor $S \mapsto \mathrm{Hom}_{\mathcal{C}}(\cdot, S)$ and $k_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^\vee$ be given by $S \mapsto \mathrm{Hom}_{\mathcal{C}}(S, \cdot)$.

Proposition 5.22

Let $\mathcal{I}$ be a small category. Use $k_{\mathcal{C}}$ and $h_{\mathcal{C}}$ we can embed $\mathcal{C}$ as subcategory of $\mathcal{C}^\vee$ and $\mathcal{C}^\wedge$, respectively.

1. *The functor $F : \mathcal{I} \to \mathcal{C}$ has a colimit if and only if the functor "$\varinjlim$"$F \in \mathcal{C}^\vee$ is representable; given the limit of F is the same as giving object $\varinjlim F \in \mathcal{C}$ together with an isomorphism $k_{\mathcal{C}}(\varinjlim F) \xrightarrow{\sim}$ "$\varinjlim$"F.*
2. *The functor $G : \mathcal{I}^{opp} \to \mathcal{C}$ has a limit if and only if "$\varprojlim$"$G \in \mathcal{C}^\wedge$ is representable; given the limit of G is the same as giving object $\varprojlim G \in \mathcal{C}$ together with isomorphism $h_{\mathcal{C}}(\varprojlim G) \xrightarrow{\sim}$ "$\varprojlim$"G.*

Thus, the property of limits is characterized by

$$\mathrm{Hom}_{\mathcal{C}}(\varinjlim F, \cdot) \xrightarrow{\sim} \varprojlim \mathrm{Hom}_{\mathcal{C}}(F(i), \cdot) = \text{“}\varinjlim\text{”} F$$

$$\mathrm{Hom}_{\mathcal{C}}(\cdot, \varprojlim G) \xrightarrow{\sim} \varprojlim \mathrm{Hom}_{\mathcal{C}}(\cdot, G(i)) = \text{“}\varprojlim\text{”} G$$

Proof. We first consider colimit. The universal property of colimit can be re-interpreted as the following: to give the colimit of F is the same as to give object $\varinjlim F$ together with arrows $\iota_i : F(i) \to \varinjlim F$, so that the morphism between functors

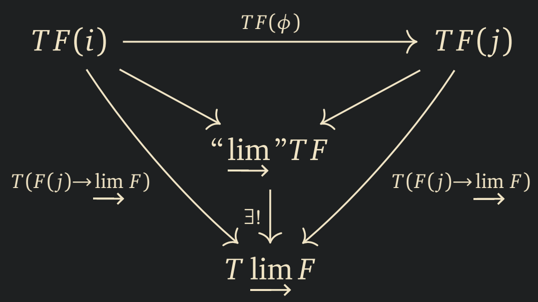
$$\begin{array}{ccccccc} k_{\mathcal{C}}(\varinjlim F) & \xleftrightarrow{=} & \mathrm{Hom}_{\mathcal{C}}(\varinjlim F, \cdot) & \xrightarrow{\xi} & \varprojlim_i \mathrm{Hom}_{\mathcal{C}}(F(i), \cdot) & \xleftrightarrow[=]{} & \text{“}\varinjlim\text{”} F \\ & & f & \longmapsto & (\iota_i^* f = f \circ \iota)_{i \in \mathcal{I}} & & \end{array}$$

is an isomorphism. Conversely, any isomorphism $\xi : k_{\mathcal{C}}(\varinjlim F) \xrightarrow{\sim}$ "$\varinjlim$"F can be induced by a family of arrows $\iota_i : F(i) \to \varinjlim F$. We can see this by Yoneda embedding. This concludes the proof, as the case for $\varprojlim$ is similar.

Now we can start to describe the relation between limits and functors. Let $T : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor, I a small category. By Proposition 5.22, we see the existence of limit is the same as representability of "$\varinjlim$"F or "$\varprojlim$"G. Thus let's now consider what's the image of limits in $\mathcal{C}_1$ under F.

First let $F : I \to \mathcal{C}_1$ and suppose the colimit $\varinjlim F$ exists. Then by Proposition 5.22 we see the "$\varinjlim$"$TF \in \mathcal{C}_2^\vee$ is the limit of TF. By universal property, we see for every

$i \xrightarrow{\phi} j$ in I, we get



and hence an arrow “$\varinjlim$”$TF \to T\varinjlim F$. We also get an arrow $T\varprojlim G \to$ “$\varprojlim$”TG in $\mathcal{C}_2^{\wedge}$, where $G : I^{\text{opp}} \to \mathcal{C}_1$, if we just flip all the arrows above.

Definition 5.23

Let T, F, G be as above, and assume the limits of F and G exists. Then we say:

1. T ***preserves*** $\varinjlim F$, if “$\varinjlim$”$(TF) \xrightarrow{\sim} T(\varinjlim F)$
2. T ***preserves*** $\varprojlim G$, if $T(\varprojlim G) \xrightarrow{\sim}$ “$\varprojlim$”(TG)

Now we are ready to prove RAPL, and in fact, also left adjoint preserves colimit:

Theorem 5.24

Consider adjoint pair (F, G, ϕ) where $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_1$. Then F preserves all $\varinjlim$ and G preserves all $\varprojlim$, if the limit exists and its small limit.

Proof. By duality of limit and colimit, we only need to show G preserves $\varprojlim$. Consider $\beta : I^{\text{opp}} \to \mathcal{C}_2$ and assume $\varprojlim \beta$ exists. Then, by Proposition 5.21, Proposition 5.21 and functoriality of ϕ, we have isomorphisms in $\mathcal{C}_1^{\wedge}$

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}_1}(\cdot, G(\varprojlim \beta)) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_2}(F(\cdot), \varprojlim \beta) \\
&= \varprojlim \mathrm{Hom}_{\mathcal{C}_2}(F(\cdot), \beta(i)) \\
&\xrightarrow{\sim} \varprojlim \mathrm{Hom}_{\mathcal{C}_1}(\cdot, G\beta(i)) \\
&= \text{“}\varprojlim\text{”} G\beta
\end{aligned}$$

It remains to verify the arrows in Definition 5.23 are isomorphisms. This is a standard exercise, but we do it anyway. For any $j \in I$ we let the projection be $p_j : \varprojlim \beta \to \beta(j)$, then we have commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}_1}(\cdot, G(\varprojlim \beta)) & \xrightarrow{(Gp_j)_*} & \mathrm{Hom}_{\mathcal{C}_1}(\cdot, G\beta(j)) \\
\downarrow & & \downarrow \phi \\
\mathrm{Hom}_{\mathcal{C}_2}(F(\cdot), \varprojlim \beta) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}_1}(F(\cdot), \beta(j)) \\
= \updownarrow & & \updownarrow = \\
\varprojlim \mathrm{Hom}_{\mathcal{C}_2}(F(\cdot), \beta(i)) & \twoheadrightarrow & \mathrm{Hom}_{\mathcal{C}_1}(F(\cdot), \beta(j)) \\
\downarrow & & \downarrow \phi^{-1} \\
\varprojlim \mathrm{Hom}_{\mathcal{C}_2}(\cdot, G\beta(i)) & \twoheadrightarrow & \mathrm{Hom}_{\mathcal{C}_1}(\cdot, G\beta(j))
\end{array}$$

where:

- the first and third square commutes by functoriality of ϕ
- the composition of the vertical arrows on the right is the identity
- the composition of the vertical arrows on the left is the isomorphism we obtained from the above computation

This concludes the isomorphism $G \varprojlim \beta \xrightarrow{\sim}$ "$\varprojlim$"$G\beta$ and the morphism $G \varprojlim \beta \to$ "$\varprojlim$"$G\beta$ in Definition 5.23 are characterized by the same family of commutative diagrams. This concludes the proof.

Of course we can ask to what extend does the converse holds? In general, this is not true, but in special cases we get the converse.

Definition 5.25

Let $\mathcal{C}$ be locally small, then we say $C \in \mathcal{C}$ is a ***cogenerator*** if the functor $h^C : \mathcal{C}^{\text{opp}} \to (\textbf{Sets})$ is faithful.

Definition 5.26

Let $\mathcal{C}$ be a locally small category, then we say $\mathcal{C}$ has a small cogenerating set, if there is a set $\{C_i\}_{i \in I}$, such that h^{C_i} are jointly faithful.

Theorem 5.27: Special Adjoint Functor Theorem

Let $\mathcal{D}$ be concrete with all limit exists, has a small cogenerating set, $G : \mathcal{D} \to \mathcal{C}$ with $\mathcal{C}$ locally small. If G preserves limit. Then G has a left adjoint.

Now we talk about completeness for categories.

Definition 5.28

A category $\mathcal{C}$ is ***complete*** if for all small categories I and all diagrams $F : I \to \mathcal{C}$, the limit $\varprojlim F$ exists. Similarly we say $\mathcal{C}$ is ***cocomplete*** if for all small I and diagram $F : I^{\text{opp}} \to \mathcal{C}$, the colimit $\varinjlim F$ exists.

Theorem 5.29: Freyd

A small category $\mathcal{C}$ is complete if and only if its induced by a partially ordered set $(P, \leq)$, such that every subset has least lower bound.

Proof. Assume $\mathcal{C}$ is complete. Assume we have distinct morphisms $f, g : X \to Y$ in $\mathcal{C}$, then for small set I we can construct $\prod_{i\in I} Y$, and hence $\operatorname{Hom}_{\mathcal{C}}(X, \prod_{i\in I} Y) \supseteq \{f, g\}^I$. Take $|I| = |\operatorname{Mor}(\mathcal{C})|$ and use the fact $|X| < |\operatorname{PowerSet}(X)|$ we see $\mathcal{C}$ must be a partially ordered set. On the other hand, we see the limits in partially ordered set is just least lower bound, and hence we are done.

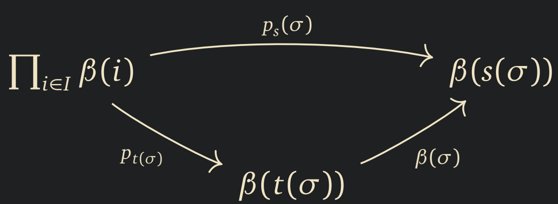
Theorem 5.30

Let I be a small category and $\mathcal{C}$ a category.

1. *If for all subsets $J \subseteq \operatorname{Mor}(I)$ and all family of objects $(X_j)_{j\in J}$ the product $\prod_{j\in J} X_j$ exists, and for all $f, g : X \to Y$ the kernel $\ker(f, g)$ exists, then all limits indexed by I exists in $\mathcal{C}$.*
2. *If for all subsets $J \subseteq \operatorname{Mor}(I)$ and $(X_j)_{j\in J}$ the coproduct $\coprod_{j\in J} X_j$ exists, and for all $f, g : X \to Y$ the cokernel $\operatorname{coker}(f, g)$ exists, then all colimits indexed by I exists in $\mathcal{C}$.*

Proof. Those two claims are dual to each other, hence we only prove the case for $\varprojlim$.

Consider $\beta : I^{\text{opp}} \to \mathcal{C}$. For morphism $\sigma : i \to j$ in I, recall we have the target and source morphisms $s(\sigma) = i$ and $t(\sigma) = j$. Construct $\prod_{i\in I} \beta(i)$ and $\prod_{\sigma\in \operatorname{Mor}(I)} \beta(s(\sigma))$. For every $\sigma \in \operatorname{Mor}(I)$ define a pair of morphisms



Thus from universal property of products we get morphisms

$$\prod_{i\in I} \beta(i) \underset{g}{\overset{f}{\rightrightarrows}} \prod_{\sigma\in \operatorname{Mor}(I)} \beta(s(\sigma))$$

. We claim the following data defines our desirede $\varprojlim \beta$:

$$\ker(f,g), \quad \left(q_j : \ker(f,g) \to \prod_{i\in I} \beta(i) \xrightarrow{p_j} \beta(j)\right) \tag{Eq. 5.1}$$

We know the limit and colimit in sets, thus by Proposition 5.21 we know the functor "$\varprojlim$"β is equivalent to

$$\begin{aligned}\varprojlim \mathrm{Hom}_{\mathcal{C}}(\cdot,\beta(i)) &= \ker\left(\prod_{i\in I}\mathrm{Hom}_{\mathcal{C}}(\cdot,\beta(i)) \rightrightarrows \prod_{\sigma\in\mathrm{Mor}(I)}\mathrm{Hom}_{\mathcal{C}}(\cdot,\beta(s(\sigma)))\right)\\ &= \mathrm{Hom}_{\mathcal{C}}(\cdot,\ker(f,g))\end{aligned}$$

This means "$\varprojlim$"β is representable. In particular, by definition of q_j we know when we take projection of $\varprojlim \mathrm{Hom}(\cdot,\beta(i)) \to \mathrm{Hom}(\cdot,\beta(j))$ is the same as taking q_{j*} : $\mathrm{Hom}(\cdot,\ker(f,g)) \to \mathrm{Hom}(\cdot,\beta(j))$. Now apply Proposition 5.22 we conclude the proof.

Corollary 5.30.1

A category is complete if and only if all equalizers and small products exists, and its cocomplete if and only if all coequalizers and small coproducts exists.

6 Flat, Projective&Injective Modules

Now we consider some application of those results. From now on until we mentioned it next time, we will assume R is commutative ring.

Example 6.1

Note $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\} \cong \mathbb{Z}/2\mathbb{Z} \coprod_{(\mathbf{Grp})} \mathbb{Z}/3\mathbb{Z}$ is the free product of $\mathbb{Z}/2\mathbb{Z}$ with $\mathbb{Z}/3\mathbb{Z}$. In particular, $\mathrm{PSL}_2(\mathbb{Z})/[\mathrm{PSL}_2(\mathbb{Z}),\mathrm{PSL}_2(\mathbb{Z})]$ has index 6, and we can see this as follows. Consider the abelianization functor $A : (\mathbf{Grp}) \to (\mathbf{Ab})$, which is a left adjoint and it preserves colimits. Thus we see

$$\begin{aligned}A(\mathrm{PSL}_2(\mathbb{Z})) &\cong A(\mathbb{Z}/2\mathbb{Z}\coprod_{(\mathbf{Grp})}\mathbb{Z}/3\mathbb{Z})\\ &= A(\mathbb{Z}/2\mathbb{Z})\coprod_{(\mathbf{Ab})}A(\mathbb{Z}/3\mathbb{Z})\\ &= \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}\end{aligned}$$

Next, we have some right exact functors:

Example 6.2

Consider the tensor product functor $(\cdot \otimes M) : (R\text{-}\mathbf{Mod}) \to (R\text{-}\mathbf{Mod})$. This functor is right exact. This is because given $A \to B \to C \to 0$, this is the same as the cokernel of $A \rightrightarrows B$ with two arrows being $A \to B$ and $A \xrightarrow{0} B$ (i.e. WLOG C can be taken to be $B/\operatorname{Im}(A)$). Thus because $(\cdot \otimes M)$ preserves colimit, we conclude

$$A \otimes M \to B \otimes M \to C \otimes M \to 0$$

In particular we also see $(\cdot \otimes M)$ commutes with direct sum (since its a coproduct), i.e. we get $(\oplus A_i) \otimes M \cong \oplus (A_i \otimes M)$.

Example 6.3

Now take the hom functor $\operatorname{Hom}_R(M, \cdot)$, and this is left exact, because kernel is a limit. Similarly we get $\operatorname{Hom}_R(M, \prod A_i) = \prod_i \operatorname{Hom}(M, A_i)$ because product is a limit.

Note the tensor in general is not left exact, and if this is the case, when we give it a name:

Definition 6.4

We say a R-module M is ***flat*** if $(\cdot \otimes M)$ is an exact functor.

Example 6.5

1. $\mathbb{Q}$ is a flat $\mathbb{Z}$-module
2. R is always flat over itself, more generally, $R_{\mathfrak{p}}$ is always flat as R-module, where $\mathfrak{p} \in \operatorname{Spec} R$.

Definition 6.6

We say R-module M is ***faithfully flat*** if $0 \to A \to B \to C \to 0$ is exact if and only if $0 \to M \otimes A \to M \otimes B \to M \otimes C \to 0$ is exact.

Example 6.7

$\mathbb{Q}$ is not faithfully flat. To see this, let A be a finite torsion abelian group. Then $A \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Indeed, $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ because $[a] \otimes \frac{b}{c} = [a] \otimes \frac{nb}{nc} = [na] \otimes \frac{b}{c} = 0$ for all $[a] \in \mathbb{Z}/n\mathbb{Z}$ and $b/c \in \mathbb{Q}$. But then any finite abelian group is given by

$$A \cong \bigoplus \mathbb{Z}/n_i\mathbb{Z}$$

for some n_i, and thus $A \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. At this point, consider an sequence of finite abelian groups $0 \to A \to B \to C \to 0$ that is not exact, and tensor with $\mathbb{Q}$ we get $0 \to 0 \to 0 \to 0 \to 0$ which has to be exact. This shows $\mathbb{Q}$ is not faithfully flat.

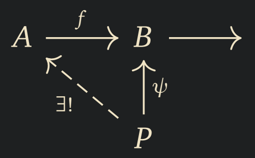
Example 6.8

The covariant hom functor is not always right exact. Indeed, consider $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$ and $M = \mathbb{Z}/2\mathbb{Z}$, then we get

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$$

is the same as $0 \to \mathbb{Z}/2\mathbb{Z}$ which cannot be exact.

A natural question to ask is when $\mathrm{Hom}_R(P, -)$ is exact. We need $A \xrightarrow{f} B \to 0$ is exact, which implies $\mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B) \to 0$ must be exact, where $(\psi : P \to A) \mapsto (f \circ \psi)$. This means that, whenever we have exact sequence $A \to B \to 0$ and a arrow $P \to B$, there must exists an unique arrow $P \to A$ that makes the following diagram commute:



Definition 6.9

An R-module is ***projective*** if $\mathrm{Hom}(P, \cdot)$ is exact.

We note the universal property of projective modules is the passage above the definition. As an immediate corollary, we see if $\{P_i\}$ is a family of projective modules, then $\bigoplus P_i$ is projective. Also, by the example above we see $\mathbb{Z}/2\mathbb{Z}$ cannot be projective.

On the other hand, we have two contravariant hom functor $\mathrm{Hom}_R(\cdot, M)$. Namely,

$$\mathrm{Hom}_R(\cdot, M)_1 : (R\textbf{-Mod}) \to (R\textbf{-Mod})^{\mathrm{opp}}$$

$$\mathrm{Hom}_R(\cdot, M)_2 : (R\textbf{-Mod})^{\mathrm{opp}} \to (R\textbf{-Mod})$$

The first hom functor is left adjoint and the second one is right exact.

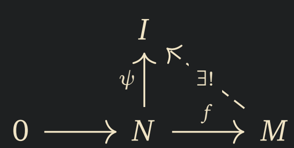
Definition 6.10

An R-module I is ***injective*** if $\mathrm{Hom}(-, I)_1$ is exact.

Well, when is this happening? Let $0 \to N \xrightarrow{f} M$ be exact, then

$$\mathrm{Hom}(M, I) \to \mathrm{Hom}(N, I) \to 0$$

under the map $\psi \mapsto \psi \circ f$ is exact. Well, this is the same as saying, we have the following diagram



At this point, we will no longer consider commutative ring R, but instead general associative ring R with a 1.

In this case, we have (R-**Mod**), (**Mod**-R) and (S-**Mod**-R), which are the left, right R-modules and (S,R)-bimodules, respectively (bimodules means $s \cdot m \cdot r = (sm) \cdot r = s \cdot (m \cdot r)$). In particular, we have (R-**Mod**-$\mathbb{Z}$) = (R-**Mod**) and similarly ($\mathbb{Z}$-**Mod**-R) = (**Mod**-R).

Now we first define tensor product for non-commutative rings.

Let $M \in$ (S-**Mod**-R) and $N \in$ (R-**Mod**-T), then we will define $M \otimes_R N \in$ (S-**Mod-T**), as follows: $M \otimes_R N$ is the free $\mathbb{Z}$-module on all symbols $e_{(m,n)}$ with $m \in M$ and $n \in N$, subject to the equivalence relation generated by

$$e_{(m_1+m_2,n)} - e_{m_1,n} - e_{m_2,n}$$

$$e_{m,n_1+n_2} - e_{m,n_1} - e_{m,n_2}$$

$$e_{mr,n} - e_{m,rn}$$

with $m, m_i \in M, n, n_i \in N$ and $r \in R$.

This still has the desired universal property that

$$\begin{array}{ccc} M \times N & \longrightarrow & P \\ \downarrow & \overset{\exists!}{\nearrow} & \\ M \otimes_R N & & \end{array}$$

where $M \times N \to P$ is R-bilinear and $M \times N \to M \otimes_R N$ is given by $(m,n) \mapsto m \otimes n$.

Example 6.11

Let $R = M_2(\mathbb{C})$ and $V = \{\begin{bmatrix} a \\ b \end{bmatrix} : a, b \in \mathbb{C}\}$. Then V is $(R,\mathbb{C})$-bimodule. Also let $W = \{\begin{bmatrix} c & d \end{bmatrix} : c, d \in \mathbb{C}\}$. Then W is $(\mathbb{C},R)$-bimodule. One can in fact compute $W \otimes_R V \cong \mathbb{C}$.

Just like tensor product, if R is not commutative, then the Hom also run into problem. Let's say ${}_RM_S$ and ${}_RN_T$, where we use ${}_XM_Y$ to mean M is left X-module and right Y-module. Then we see

$$\mathrm{Hom}_{(R\text{-}\mathbf{Mod})}(M,N) \in (S\text{-}\mathbf{Mod}\text{-}T)$$

The structure is given by the following: let $\phi \in \mathrm{Hom}_{(R\text{-}\mathbf{Mod})}(M,N)$, then

$$(s \cdot \phi \cdot t)(m) := \phi(m \cdot s) \cdot t$$

Now we describe the tensor hom adjunction. Suppose we have ${}_SM_R$, ${}_RN_T$ and ${}_SP_U$. Then we want to capture the isomorhpism $P^{M\times N} \cong (P^M)^N$. Thus, what we want is

$$\mathrm{Hom}(M \otimes N, P) \cong \mathrm{Hom}(N, \mathrm{Hom}(M,P))$$

Let's figure out what the categories are. Well, $M \otimes_R N$ is (S,T)-module, and P is (S,U)-module. Hence, what we want is $\mathrm{Hom}_{(S\text{-}\mathbf{Mod})}(M \otimes N, P)$. On the other hand, N is (R,T)-module and $\mathrm{Hom}(M,P)$ is (R,U)-module, which shows we want

$$\mathrm{Hom}_{(R\text{-}\mathbf{Mod})}(N, \mathrm{Hom}_{(S\text{-}\mathbf{Mod})}(M,P))$$

Overall, the isomorphism is a (T,U)-bimodule morphism.

Then we get back to the flat, projective and injective modules. First, we say P is projective if $\mathrm{Hom}_R(P,\cdot)$ is exact, if $P \in (R\text{-}\mathbf{Mod})$.

Proposition 6.12

The following are equivalent:

1. *P is projective*
2. *all short exact sequence $0 \to A \to B \to P \to 0$ splits*
3. *there is Q such that $P \oplus Q \cong \mathbb{R}^{\oplus I}$ where I is some index set.*

Proposition 6.13

1. *If R is a commutative local ring, then f.g. projective modules are free. In fact, all projective module over commutative local ring is free.*
2. *If R is commutative PID, then projective modules are free.*
3. *If R is commutative Noetherian with $\operatorname{Spec} R$ is connected (i.e. $0, 1$ are the only idempotents), then every non-f.g. projective module is free.*

Definition 6.14

A projective R-module P is ***stably free*** if there exists $a \in \mathbb{N}$, such that $P \oplus R^a$ is free.

Remark 6.15: Eilenberg Swindle

Suppose $P \oplus Q = F$ with F free. Then we can do

$$(P \oplus Q) \oplus (P \oplus Q) \oplus \ldots \cong F \oplus F \oplus \ldots$$

where

$$(P \oplus Q) \oplus (P \oplus Q) \oplus \ldots \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \ldots \cong P \oplus (F \oplus F \oplus \ldots)$$

where at the end, $\bigoplus F$ is free.

Example 6.16: Swan's Example

We will construct something thats projective stably free but not free.

We will construct P to be a projective stably free rank 2 module over

$$A = \mathbb{R}[X,Y,Z]/(1 - X^2 - Y^2 - Z^2)$$

such that $P \oplus A \cong A^3$ and $P \ncong A^2$. Set $I = (1 - X^2 - Y^2 - Z^2)$ and let x, y, z be the image of X, Y, Z in A, i.e. $R = \mathbb{R}[x, y, z]$ such that $x^2 + y^2 + z^2 = 1$. Consider $\phi : A^3 \to A$ given by $(a, b, c) \mapsto ax + by + cz$. Observe since $x^2 + y^2 + z^2 = 1$ we see $(ux, uy, uz) \mapsto u$ under ϕ. This makes A projective because its free. Now consider

$$0 \to \ker\phi \to A^3 \to A \to 0$$

This splits and so $P := \ker\phi$ is stably free, as $P \oplus A = A^3$.

We claim P is not free. To show this, it suffices to show $Q := P \otimes_A S$ is not free, where $A \hookrightarrow S := C(\mathbb{S}^2; \mathbb{R})$ is the embedding of the polynomial real valued functions on the 2-sphere $\mathbb{S}^2$ into all continuous real valued functions on 2-sphere. Because $0 \to P \to A^3 \to A \to 0$ splits, we see

$$0 \to Q \to S^3 \to S \to 0$$

splits by the map α, which sends $(a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}, c\frac{\partial}{\partial z}) \mapsto ax + by + cz$.

Now we invoke the following theorem: if X is compact Hausdorff, then the category of fintie rank real vector bundles of X is equivalent to the category of fintie rank projective $C(X; \mathbb{R})$-modules. The equivalence is given by $\mathscr{E} \mapsto \Gamma(X, \mathscr{E})$. Under this equivalence, we see the above sequence becomes

$$0 \to \mathscr{T} \to \mathscr{O}\frac{\partial}{\partial x} \oplus \mathscr{O}\frac{\partial}{\partial y} \oplus \mathscr{O}\frac{\partial}{\partial z} \xrightarrow{\alpha} \mathscr{O} \to 0$$

of vector bundles over $\mathbb{S}^2$, where $\mathscr{O}$ is the trivial vector bundle. However, this is exactly the short exact sequence that defines the tangent bundle on $\mathbb{S}^2$. Thus we see Q is free over S if and only if the tangent bundle on $\mathbb{S}^2$ is trivial. But by Hairy Ball theorem we see that the tangent bundle on $\mathbb{S}^2$ does not admit any nowhere vanishing section, i.e. its not trivial.

Next, we will work towards Lazard theorem, which says flat R-modules is the same as filtered limits of free modules. In practice, this is very nice because it means we can reduce to the free case when we look at properties for flat modules.

First, we will need to generalize the notion of filtered limits.

Definition 6.17

A category $\mathcal{B}$ is ***filtered*** if every finite subcategory has a cocone.

This really means that when we have a finite set of objects $\{B_1, ..., B_k\}$, then we can find B with $B_i \to B$, and whenever we have $f, g : B_1 \to B_2$ we get a diagram

$$B_1 \overset{f}{\underset{g}{\rightrightarrows}} B_2 \longrightarrow B$$

When we take a colimit of a filtered category, this is really like taking a filtered colimit. Indeed, if $\{M_i\}$ is a filtered subcategory of R-modules, then

$$\varinjlim M_i = \coprod M_i / \sim$$

where $m_i \in M_i$ and $m_j \in M_j$ are equivalent iff we can find $f : M_i \to M$ and $g : M_j \to M$ such that $f(m_i) = g(m_j)$. This is obviously an equivalence relation. Also, we see this is R-module, where we define $[m_i + m_j]$ by $[f_i(m_i) + f_j(m_j)]$, where $m_i \in M_i$, $m_j \in M_j$, and the f_i, f_j exists by the definition of filtered category. You should check this is well-defined.

We now proceed to prove the following result:

Theorem 6.18: Govorov-Lazard

An A-module M is flat iff M is a filtered colimit of finite rank free A-modules.

Here the filtered/direct colimit assumption is necessary. If G is the colimit of the diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} \\ \downarrow{\scriptstyle \times 2} & & \\ \mathbb{Z} & & \end{array}$$

Then $G \cong \mathbb{Z}^3/((1,0,-2),(1,-2,0))$ has torsion, and thus cannot be flat $\mathbb{Z}$-module. Indeed. observe

$$2(1,-1,-1) = (1,0,-2) + (1,-2,0) = 0$$

but $(1,-1,-1) \neq 0$ since if

$$(1,-1,-1) = a(1,0,-2) + b(1,-2,0)$$

for some $a, b \in \mathbb{Z}$, then $1 = 2a = 2b$, which is impossible.

Lemma 6.19

Let M be R-module. The following are equivalent:

1. *M is flat over R*
2. *for every injection of R-modules $N \subseteq N'$, the map $N \otimes_R M \to N' \otimes_R M$ is injective*
3. *for every ideal $I \subseteq R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective*
4. *for every finitely generated ideal I of R, the map $I \otimes_R M \to R \otimes_R M = M$ is injective*

Proof. It suffices to show (4) $\Rightarrow$ (1). Say $N_1 \to N_2 \to N_3$ is exact. Let $K = \ker(N_2 \to N_3)$ and $Q = \operatorname{im}(N_2 \to N_3)$. Then we get maps

$$N_1 \otimes_R M \to K \otimes_R M \to N_2 \otimes_R M \to Q \otimes_R M \to N_3 \otimes_R M$$

Observe the first and third arrow are surjective. Thus if we show the second and fourth are injective, then we are done. Indeed, assume the second and fourth arrows are injective, then sicne tensor is right exact, let's tensor $K \to N_2 \to Q \to 0$ by M and we get

$$K \otimes M \to N_2 \otimes M \to Q \otimes M \to 0$$

is exact. Thus $\ker(N_2 \otimes M \to Q \otimes M) = \operatorname{im}(K \otimes M \to N_2 \otimes M)$. Since $\operatorname{im}(K \otimes M \to N_2 \otimes M) = \operatorname{im}(N_1 \otimes M \to N_2 \otimes M)$ (by surjectivity of $N_1 \otimes M \to K \otimes M$) and $\ker(N_2 \otimes M \to$

$Q \otimes M) = \ker(N_2 \otimes M \to N_3 \otimes M)$ (by injectivity of $Q \otimes M \to N_3 \otimes M$), this becomes $\ker(N_2 \otimes M \to N_3 \otimes M) = \operatorname{im}(N_1 \otimes M \to N_2 \otimes M)$, which shows the functor $- \otimes_R M$ is exact, i.e. M is flat.

Thus, it suffices to show the second and fourth arrows are injective. Hence, it suffices to show $- \otimes_R M$ transforms injective R-modules maps into injective R-module maps.

Assume $K \to N$ is injective R-module map and $x \in \ker(K \otimes M \to N \otimes M)$. We have to show x is zero. The R-module K is the union of its finite R-submodules; hence, $K \otimes M$ is the colimit of R-modules of the form $K_i \otimes M$ where K_i runs over all finite R-submodules of K (because tensor product commutes with colimits). Thus, for some i our x comes from an element $x_i \in K_i \otimes M$. Thus we may assume K is finite R-module, and thus regard the injection $K \to N$ as an inclusion, i.e. $K \subseteq N$.

The R-module N is the union of its finite R-submodules that contains K. Hence, $N \otimes M$ is the colimit of R-modules of the form $N_i \otimes M$ where N_i are fintie submodules of N that contains K. This is a colimit over a directed system, hence we see the element $x \in K \otimes M$ maps to zero in at least one of these R-modules $N_i \otimes M$ (since x maps to zero in $N \otimes M$). Thus we may assume N is finite R-module.

Assume N is finite. Write $N = R^{\oplus n}/L$ and $K = L'/L$ for some $L \subseteq L' \subseteq R^{\oplus}$. For any R-submodule $G \subseteq R^{\oplus n}$, we have a canonical map $G \otimes M \to M^{\oplus n}$ obtained by composign $G \otimes M \to R^n \otimes M = M^{\oplus n}$. It suffices to prove $L \otimes M \to M^{\oplus n}$ and $L' \otimes M \to M^{\oplus n}$ are injective. Indeed, if those two are injective, then

$$K \otimes M = L' \otimes M / L \otimes M \to M^{\oplus n} / L \otimes M$$

is also injective.

Thus, it suffices to show that $L \otimes M \to M^{\oplus n}$ is injective when $L \subseteq R^{\oplus n}$ is an R-submodule. We do this by induction on n. The base case $n = 1$ we handle below. Let us do the induction step first. Assume $n > 1$ and set $L' = L \cap R \oplus 0^{\oplus(n-1)}$. Then $L'' = L/L'$ is a submodule of $R^{\oplus(n-1)}$. We obtain a diagram

$$\begin{array}{ccccccccc} & & L' \otimes M & \longrightarrow & L \otimes M & \longrightarrow & L'' \otimes M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & M^{\oplus n} & \longrightarrow & M^{\oplus n-1} & & \end{array}$$

By induction and base case the left and right vertical arrows are injective. The row is exact. Thus the middle vertical arrow is injective as well, and we are done.

It remains to show the base case. Let $L \subseteq R$ be R-submodule, i.e. $L = I$ for some ideal I, and we have to show $I \otimes_R M \to M$ is injective for any ideal I of R. We konw this is true when I is finitely generated. But $I = \bigcup I_\alpha$ is a union of f.g. ideals I_α, and so $I = \varinjlim I_\alpha$. But $\otimes$ commutes with colimits, and all $I_\alpha \otimes M \to M$ are injective by assumption, we are done.

Lemma 6.20: Equational Criterion For Flatness

M is a flat A-module iff any relation $\sum_{i=1}^n f_i m_i = 0$, $f_i \in A$, $m_i \in M$ is trivial, i.e. we can find $m_j' \in M$ and $a_{ij} \in A$ such that

$$m_i = \sum_j a_{ij} m_j' \quad \textit{and} \quad \sum_i a_{ij} f_i = 0$$

In other word, we can write $0 = \sum_{i=1}^n f_i m_i$ using zero coefficients,

$$\sum_j \left(\sum_i a_{ij} f_i \right) m_j' = \sum_j 0 \cdot m_j' = 0$$

Proof. ($\Rightarrow$): Suppose M is flat, $\sum_i f_i m_i = 0$ in M, and let $I = (f_i)_i \subseteq A$ be the ideal generated by the coefficients. The exact sequence $0 \to I \to A$ remains exact after tensoring with M, so $\sum_i f_i \otimes m_i = 0$ in $I \otimes M$. If $\phi : A^n \to I$ is defined by $\phi(a_1, \ldots, a_n) = \sum_i a_i f_i$ and K is its kernel, the then exact sequence $0 \to K \to A^n \xrightarrow{\phi} I \to 0$ remains exact after tensoring with M, so there is $\sum_j k_j \otimes m_i'$ in $K \otimes M$ such that $\sum_j k_j \otimes m_j' = \sum_i e_i \otimes m_i$ (where $\{e_i\}$ is the standard basis for A^n), since $(\phi \otimes 1)(\sum_i e_i \otimes m_i) = 0$. Each k_j can be expressed as $k_j = \sum_i a_{ij} e_i$ so that

$$\sum_i e_i \otimes \left(\sum_j a_{ij} m_j' \right) = \sum_i e_i \otimes m_i \in A^n \otimes M \cong M^n$$

and $m_i = \sum_j a_{ij} m_j'$ (the first condition above). Also, $k_j \in K = \ker(\phi)$ so $\phi(k_j) = 0 = \sum_i a_{ij} f_i$ (the second condition above). This proves the ($\Rightarrow$) direction.

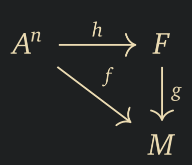
($\Leftarrow$): Suppose every relation in M is trivial. Let $I \xrightarrow{\iota} A$ be the inclusion of a finitely generated ideal. Recall flatness of M is equivalent to injectivity of $I \otimes M \to A \otimes M$ for any such I. If $(\iota \otimes 1)(\sum_i f_i \otimes m_i) = 0 \in A \otimes M$ then $\sum_i f_i m_i$ is a relation in M, hence trivial. Using the notation of the previous lemma, we have

$$\sum_i f_i \otimes m_i = \sum_i f_i \otimes \left(\sum_j a_{ij} m_j' \right) = \sum_j \left(\sum_i f_i a_{ij} \right) \otimes m_j' = 0$$

and $\iota \otimes 1$ is injective.

Corollary 6.20.1

M is a flat A-module iff whenever given a map $f : A^n \to M$ and a finitely generated $N \le \ker(f)$, there is a factorization



with F a finite rank free A-module and $N \leq \ker(h)$.

Proof. If N is generated by one element $x = (f_i)_i$, $f(x) = \sum_i f_i m_i = 0$, then M is flat iff this relation is trivial $m_i = \sum_j a_{ij} m'_j$, $\sum_i a_{ij} f_i = 0$, by the above Lemma. Take $F = A^m$, $h(a_1, \ldots, a_n) = \sum_{i,j} a_{ij} a_i e_j$ where $\{e_j\}$ is the standard basis, and $g(a_1, \ldots, a_m) = \sum_j a_i m'_j$. Then we see

$$h(f_1, \ldots, f_n) = \sum_{i,j} a_{ij} f_i e_j = 0$$

$$(g \circ h)(a_1, \ldots, a_n) = g\left(\sum_{i,j} a_{ij} a_i e_j\right) = \sum_{i,j} a_{ij} a_i m'_j = \sum_i a_i m_i = f(a_1, \ldots, a_n)$$

Now if $N = N' + Ax'$ and $A^n \xrightarrow{h'} F' \xrightarrow{g'} M$ factors f with $N' \leq \ker(h')$, then we can find F, h'' and g with $h'(x) \in \ker(h'')$ such that the following commutes

$$\begin{array}{ccccc} A^n & \xrightarrow{h'} & F' & \xrightarrow{h''} & F \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g'} & \swarrow{\scriptstyle g} & \\ & & M & & \end{array}$$

Taking $h = h'' \circ h'$ we have $N \leq \ker(h)$ and $f = g \circ h$.

Next, recall that given filtered systems $\{M'_i\}, \{M_i\}$ and $\{M''_i\}$, and arrows $f_i : M'_i \to M_i$, $g_i : M_i \to M''_i$, if all the sequences

$$M'_i \to M_i \to M''_i$$

are exact then

$$\varinjlim M'_i \to \varinjlim M_i \to \varinjlim M''_i$$

is also exact.

Definition 6.21

An induced sub-poset $(J, \leq)$ of $(I, \leq)$ is ***cofinal*** in $(I, \leq)$ if for any $i \in I$, there is $j \in J$ so $i \leq j$.

Lemma 6.22

If J is cofinal in the directed poset I and $\{M_i\}$ is a directed system of A-modules, then

$$\varinjlim_{i \in I} M_i = \varinjlim_{j \in J} M_j$$

Lemma 6.23

Every A-module M is a filtered colimit of finitely presented A-modules.

Proof. Take an exact sequence $0 \to K \to A^I \to M \to 0$ and let

$$\Lambda := \{(J,N) : J \subseteq I \text{ finite }, N \subseteq K \cap A^J \text{ f.g.}\}$$

partially ordered by $(J',N') \leq (J,N)$ if $J' \subseteq J$ and $N' \subseteq N$. For $\lambda = (J,N)$ define

$$M_\lambda = A^J/N$$

One checks this is a filtered system, and we have $h : \varinjlim_\lambda M_\lambda \to M$. This is what we want.

Now we are ready to prove Lazard's theorem.

Proof of Theorem 6.18. Suppose M is flat A-module. Consider $I = M \times \mathbb{Z}$ in the aboce lemma, and $A^I \xrightarrow{q} M$ by $1_{(m,v)} \mapsto m$ (the generator of the ith coordinate gets mapped to the projection of $i = (m,v)$ onto M). Let $\lambda = (J,N) \in \Lambda$ with $J \subseteq I$ a finite set and $N \leq \ker(f) \cap A^J$ a f.g. module, then by the Corollary 6.20.1, the map $M_\lambda := A^J/N \xrightarrow{q_\lambda} M$ factors through a finite rank free A-module F

$$M_\lambda \xrightarrow{\bar{h}} F \xrightarrow{g} M, \quad h : A^J \to F$$

such that $q_\lambda = g \circ \bar{h}$. We now realize this F as M_μ for some $\lambda \leq \mu$, giving a cofinal subset of Λ consisting of finite rank free A-modules.

Let $\{b_1, \dots, b_n\}$ be a basis for F and choose $i_1, \dots, i_n \in I$ such that $i_l \notin J$ and

$$q(1_{i_l}) = g(b_l), \quad q : A^I \to M, \quad g; F \to M$$

This is possible since I is M times $\mathbb{Z}$. Let $J' = J \cup \{i_1, \dots, i_n\}$ and define a map $A^{J'} \xrightarrow{\tilde{h}} F$ extending h and mapping 1_{i_l} to b_l, with kernel N'. The following diagram commutes

$$\begin{array}{ccc} A^{J'} & \xrightarrow{\tilde{h}} & F \\ \downarrow & & \downarrow g \\ A^I & \xrightarrow{q} & M \end{array}$$

so $N' \leq \ker(q)$. The top arrow splits as its a surjection to free objects. Thus $A^{J'} \cong N' \oplus F$ and N' is f.g. as well. Thus $\mu = (J', N') \in \Lambda$ and $\lambda \leq \mu$. Thus we see M is indeed the filtered colimit of finite rank free modules, as desired.

7 Abelian Category

Abelian categories should be something similar to the category of R-modules.

Definition 7.1

A ***preadditive category*** $\mathcal{C}$ is one where each Hom-set is an abelian group.

In particular, given preadditive category, we should have $0_{A,B} \in \mathrm{Hom}(A,B)$. Next, suppose we have $A \xrightarrow{g} B \xrightarrow{f} C$, then we want the composition map $(f,g) \mapsto f \circ g$ to be $\mathbb{Z}$-linear. That is, we want $(f, g_1 + ng_2) \mapsto f \circ (g_1 + ng_2) = f \circ g_1 + nf \circ g_2$ for non-negative integer $n \geq 0$. Similarly we want $(f_1 + nf_2) \circ g = f_1 \circ g + nf_2 \circ g$.

Definition 7.2

A ***additive category*** is a preadditive category where all (includes empty) finite products and coproducts exists.

In fact, we will see, in an additive category, since we have empty product and coproduct, we have initial and terminal objects. In particular, the initial and terminal object will be isomorphic, and we just call it the 0 object.

Indeed, let I be initial and T be terminal, then we have

$$I \xrightarrow{\exists!\alpha} T \xrightarrow{0_{T,I}} I$$

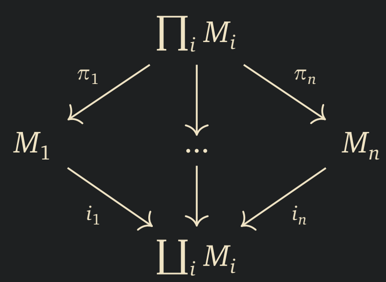
where $0_{T,I}$ exists as the category is preadditive. However, since we have unique $I \to I$, i.e. the Id_I, this means $\mathrm{Id}_I = 0_{I,I} = 0_{T,I} \circ \alpha$, i.e. $0_{I,I} = \mathrm{Id}_I$. However, α must be $0_{I,T}$. Similarly we see $0_{T,T} = \mathrm{Id}_T$. Thus we see $0_{I,T} \circ 0_{T,I} = \mathrm{Id}_I$ and $0_{T,I} \circ 0_{I,T} = \mathrm{Id}_T$ and $I \cong T$. This in fact is true for a more general setting.

Theorem 7.3

Let $\mathcal{C}$ be an additive category. Then

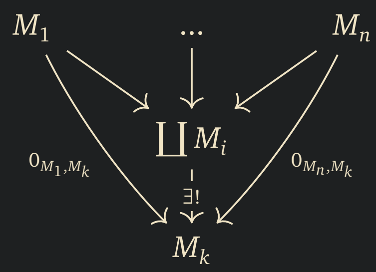
$$\prod_{i=1}^{n} M_i \cong \coprod_{i=1}^{n} M_i$$

Proof. We have the following diagram



where the jth composition is given by $(m_i)_{i=1}^n \mapsto m_j \mapsto (0, \dots, m_j, \dots, 0)$. We claim $\theta := \sum_k i_k \circ \pi_k : \prod M_i \to \coprod M_i$ is an isomorphism.

We need to produce a map $\psi : \coprod M_i \to \prod M_i$. However, we have cocone



Thus we can find unique $\beta_k : \coprod M_i \to M_k$, such that

$$\beta_k \circ i_j = \delta_{ik} := \begin{cases} \mathrm{Id}_{M_k} & \text{if } j = k \\ 0_{M_j,M_k} & \text{otherwise} \end{cases}$$

where δ_{ik} is the Kronecker delta. We can revert all the arrows in the above diagram, and we can get $\gamma_k : M_k \to \prod M_i$, such that

$$\pi_i \circ \gamma_k = \delta_{ik}$$

Define

$$\psi := \sum_k \gamma_k \circ \beta_k$$

It remains to prove θ and ψ are inverse of each other. Will,

$$\psi \circ \theta = \sum_{k,l} \gamma_k \circ (\beta_k \circ i_l) \circ \pi_l = \sum_{k,l} \gamma_k \circ (\delta_{k,l}) \circ \pi_l = \sum_{k=1}^{n} \gamma_k \circ \pi_k = \mathrm{Id}_{\prod M_i}$$

where $\sum_{k=1}^{n} \gamma_k \circ \pi_k = \mathrm{Id}_{\prod M_i}$ is obtained by using universal property of product.

Definition 7.4

Let $\mathcal{C}$ be a additive category, and $f : A \to B$ an arrow in $\mathcal{C}$, then we define

$$\ker(f) := \mathrm{Eq}(f, 0)$$

to be the equalizer of the diagram $A \underset{0}{\overset{f}{\rightrightarrows}} B$. Dually, we define the cokernel of $g : B \to C$ be the coequalizer of $B \underset{0}{\overset{g}{\rightrightarrows}} C$.

Definition 7.5

A ***preabelian category*** is an additive category where all kernels and cokernels exists.

Equivalently, preabelian category is additive category where all finite limits and colimits exists.

Next, recall mono and epi morphisms are defined as follows: we say $f : A \to B$ is monomorphism if whenever we have

$$C \underset{h_2}{\overset{h_1}{\rightrightarrows}} A \longrightarrow B$$

with $fh_1 = fh_2$ then $h_1 = h_2$. Similarly, we say $f : A \to B$ is epimorphism, if whenever we have

$$A \xrightarrow{f} B \underset{h_2}{\overset{h_1}{\rightrightarrows}} C$$

with $h_1 f = h_2 f$ then $h_1 = h_2$.

Clearly mono does not imply injection, and epi does not imply surjection.

Example 7.6

In the category of Hausdorff topological space, the inclusion $\mathbb{Q} \to \mathbb{R}$ is not onto but epimorphism. On the other hand, in the category of commutative rings, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both epi and mono.

In a concrete category, injection implies mono and surjection implies epi. This is because this is true in (**Sets**), as one should check.

Remark 7.7

Let $f : A \to B$, then it is mono if and only if $\varprojlim(A \xrightarrow{f} B \xleftarrow{f} A)$ exists. Dually, $f : B \to A$ is epi iff $\varinjlim(A \xleftarrow{f} B \xrightarrow{f} A)$ exists.

Definition 7.8

A monomorphism $f : A \to B$ is ***normal*** if it is a kernel of some diagram, i.e. we have a (universal) diagram

$$A \xrightarrow{f} B \underset{0}{\overset{g}{\rightrightarrows}} C$$

Similarly, an epimorphism $g : B \to C$ is ***normal*** if it is a cokernel of some diagram.

Definition 7.9

A ***abelian*** category is a preabelian category where all monomorphisms and epimorphisms are normal.

Example 7.10

The category of abelian groups is abelian category. Indeed, say $f : A \to B$ is mono, then we see f is the kernel of $B \to B/\operatorname{im}(f)$. Similarly, if $g : B \to C$ is epimorphism, then $\ker(g) \to B \xrightarrow{g} C$ witness g being a cokernel.

The above example still holds when we move to (R-**Mod**) or (**Mod**-R).

Example 7.11

Let X be a topological space, and $\operatorname{Op}(X)$ the category of open sets by inclusion. Then a sheaf $\mathscr{O}_X$ is a functor $\operatorname{Op}(X)^{\text{opp}} \to (\textbf{AbGrp})$, and we say $(X, \mathscr{O}_X)$ is a ringed space if $\mathscr{O}_X : \operatorname{Op}(X)^{\text{opp}} \to (\textbf{Ring})$.

For example, take $X = \operatorname{Spec} R = \{\mathfrak{p} : \mathfrak{p} \text{ prime ideal of } R\}$ with R integral domain, and $\mathscr{O}_X$ the structure sheaf, i.e.

$$\mathscr{O}_X(U) = \{\frac{f}{g} : f, g \in R, g \text{ non-zero on } U\}$$

Here we say g is non-zero on U if the stalk of g at $u \in U$ is not equal 0 for all u. Here, if $u = \mathfrak{p}$, then in fact we have the stalk at u just equal $R_{\mathfrak{p}}$, the localization at prime $\mathfrak{p}$.

Next, given ringed spaces $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$, a morphism between them will be a pair $(f, f^\sharp)$ where $f : X \to Y$ continuous, and $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a natural transformation where $f_*\mathscr{O}_X : \operatorname{Op}(Y)^{\text{opp}} \to (\textbf{Ring})$ is given by $f_*\mathscr{O}_X(V) := \mathscr{O}_X(f^{-1}(V))$ for $V \subseteq Y$ open.

More generally, given ringed space $(X, \mathscr{O}_X)$, we can consider the category of sheaves of $\mathscr{O}_X$-modules. This is just globalization of modules on $\mathscr{O}_X(U)$. This is an example of abelian category. To see this, all things are trivial. To see monomorphisms and epimorphisms are normal, we just note we can globalize the kernel and cokernel construction, e.g. given a morphism of sheaves of $\mathscr{O}_X$-modules $\mathscr{F} \to \mathscr{G}$, we can form $\operatorname{coker}(\mathscr{F} \to \mathscr{G})$ by taking sheafification of the presheaf $U \mapsto \operatorname{coker}(\mathscr{F}(U) \to \mathscr{G}(U))$. Then mono and epi are normal follows by the fact mono and epi in (R-**Mod**) are normal.

Our next topic is Mitchell's embedding theorem. What this says is that, given $\mathcal{A}$ an abelian category, $\{A_i\}_i$ a set of objects in $\mathcal{A}$. Then there exists a small abelian subset $\mathcal{A}_0$ of $\mathcal{A}$ that contains $\{A_i\}$ and an embedding $\mathcal{A}_0 \hookrightarrow$ (R-**Mod**).

This is very nice, because we can now check mono and epi for any abelian category with ease.

Example 7.12

We claim $\mathcal{A}$ is an abelian category and $f : A \to B$ is both mono and epi, then f is an isomorphism. Indeed, we can create $\mathcal{A}_0$ which contains A, B, and embed $\mathcal{A}_0$ to some (R-**Mod**) by fully faithful F. Then clearly in (R-**Mod**) we have Ff is mono and epi, but in (R-**Mod**) it is kind of obvious mono+epi implies isomorphism. Thus we are done.

The proof of the following result will be given in a separate section.

Theorem 7.13: Mitchell Embedding

Let $\mathcal{A}$ be a small abelian category. Then there is a fully faithful exact functor

$$F : \mathcal{A} \to (R\text{-}\mathbf{Mod})$$

for some ring R.

Here, given $F : \mathcal{C} \to \mathcal{D}$. Then full functor means F is onto on the hom set for all $C_1, C_2 \in \mathcal{C}$, i.e. $\operatorname{Hom}(C_1, C_2) \to \operatorname{Hom}(FC_1, FC_2)$ is surjective. On the other hand, F is faithful meanings this map on the hom set is injective. Next, a functor between abelian categories is exact if F is additive (meaning the map on hom sets is group homomorphism) and maps exact sequence to exact sequence.

Its not hard to see if F is additive, then $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$, as one should check.

Next, we ask what is exactness in an abelian category. Well,

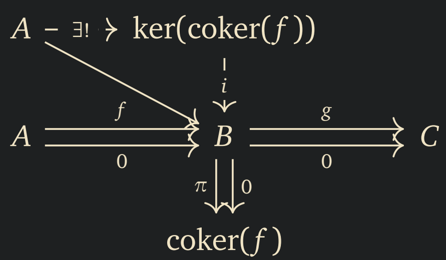
$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact at B if $g \circ f = 0_{A,C}$ and $\operatorname{im}(f) = \ker(g)$ where $\operatorname{im}(f) := \ker(\operatorname{coker}(f))$ by definition.

What is the map

$$\ker(\operatorname{coker}(f)) \to \ker(g)$$

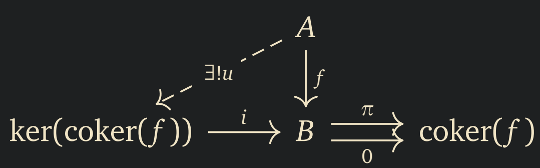
Well, note we have



Here we have unique arrow $A \to \ker(\operatorname{coker}(f))$ because we have

$$A \underset{0}{\overset{f}{\rightrightarrows}} B \xrightarrow{\pi} \operatorname{coker}(f)$$

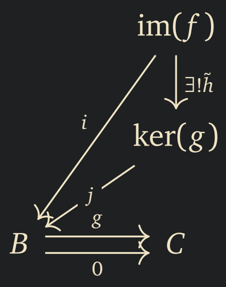
and thus



where $f = i \circ u$.

We remark $u : A \to \operatorname{im}(f)$ is an epimorphism.

Now get back to exactness, note $g \circ f = 0$, thus $g \circ i \circ u = 0 = 0 \circ u$. Since u is an epimorphism, we see this means $g \circ i = 0$. We can see this by the cone



To use Mitchell embedding for any abelian category, we will need the following lemma.

Lemma 7.14

Let $\mathcal{A}$ be an abelian category and $\{A_i\}$ be a set of objects of $\mathcal{A}$. Then there exists a full small abelian subcategory $\mathcal{A}_0$ of $\mathcal{A}$, such that $\{A_i\}$ are objects of $\mathcal{A}_0$.

Proof. Given $X \subseteq \mathcal{A}$. Let $\mathcal{C}(X)$ be the full subcategory of $\mathcal{A}$ whose objects includes:

1. X
2. isomorphism class representatives of all finite limits/colimits
3. isomorphism class representatives of witnesses that monos/epis are normal

Note $\mathcal{C}(X)$ need not be abelian. Thus define $\mathcal{C}^2(X) = \mathcal{C}(\mathcal{C}(X)), \ldots, \mathcal{C}^n(X) = \mathcal{C}(\mathcal{C}^{n-1}(X))$ and so on. Take

$$\mathcal{A}_0 = \bigcup_{n \geq 1} \mathcal{C}^n(X)$$

With this lemma, when we are working with $\{A_i\} \subseteq \mathcal{A}$, we can first take $\mathcal{A}_0$ as above, then embed $\mathcal{A}_0$ into some R-modules, and be happy about it.

Remark 7.15

If $\mathcal{A}, \mathcal{B}$ are abelian categories, and $F : \mathcal{A} \to \mathcal{B}$ is fully faithful exact, then $A \xrightarrow{u} B$ is mono in $\mathcal{A}$ if and only if $F(u)$ is monomorphism in $\mathcal{B}$. A similar claim holds for epimorphisms.

To see this, we need a fact, which says u is mono iff $\ker(u) = 0$ iff $0 \to A \xrightarrow{u} B$ is exact. Similarly we have the fact that v is epi iff $\operatorname{coker}(v) = 0$ iff $B \xrightarrow{v} C \to 0$ is exact.

The proof is now left as an exercise.

8 Proof of Mitchell Embedding

To prove the Mitchell embedding, we are going to first embed $\mathcal{A}$ into $\mathrm{Pro}(\mathcal{A})$ (or dually into $\mathrm{Ind}(\mathcal{A})$), then take $R = \mathrm{End}(\mathrm{Pro}(\mathcal{A}))(S)$ where S is a set of projective generators.

To make sense of this, we first introduce Grothendieck categories.

Definition 8.1

Let $\mathcal{E}$ be a category, Σ a non-empty set in $\mathcal{E}$. Then:

- We say Σ is a ***set of generators***, if, for any pair of arrows $f, g : x \to y$, if for all $s \in \Sigma$ and $\epsilon : s \to x$ we have $f\epsilon = g\epsilon$, then $f = g$. If Σ is a set of generators, then we say $s \in \Sigma$ is a ***generator***.
- We say Σ is a ***set of cogenerators***, if, for any pair of arrows $f, g : x \to y$, if for all $s \in \Sigma$ and all $\delta : y \to s$, we have $\delta f = \delta g$, then $f = g$. If $s \in \Sigma$ with Σ a set of cogenerators, then we say s is a ***cogenerator***.

Definition 8.2

Let X be an object of abelian category $\mathcal{A}$. Then we say X is ***projective*** if $\mathrm{Hom}(X, \cdot) : \mathcal{A} \to (\mathbf{AbGrp})$ is an exact functor. Dually, we say X is ***injective*** if $\mathrm{Hom}(\cdot, X) : \mathcal{A}^{\mathrm{opp}} \to (\mathbf{AbGrp})$ is an exact functor.

Proposition 8.3

Let $\mathcal{A}$ be abelian, $X \in \mathcal{A}$ be injective (resp. projective). If for all $T \in \mathcal{A}$ with $T \neq 0$, $\mathrm{Hom}(T, X) \neq 0$ (resp. $\mathrm{Hom}(X, T) \neq 0$), then X is a cogenerator (resp. generator)

Proof. We only do the injective case. If X is cogenerator, then apply the definition of cogenerator to $T \overset{\mathrm{Id}_T}{\underset{0}{\rightrightarrows}} T$ we get $\delta \in \mathrm{Hom}(T, X)$ such that $\delta = \delta \circ \mathrm{Id}_T \neq \delta \circ 0 = 0$.

Conversely, we want to show that, if $h : S \to T$ is non-zero then there exists $\delta \in \mathrm{Hom}(T, X)$ such that $\delta h \neq 0$. Observe we can find $\delta' \in \mathrm{Hom}(\mathrm{im}(h), X)$ such that $\delta' \neq 0$. But by property of epimorphism we see the composition $S \twoheadrightarrow \mathrm{im}(h) \xrightarrow{\delta'} X$ is non-zero. Now apply the property of injective objects we see we can extend δ' to $\delta : T \to X$.

Definition 8.4

Let $\mathcal{A}$ be abelian category. Then:

1. If for all $X \in \mathcal{A}$, there is injective object I and monomorphism $X \to I$, then we say $\mathcal{A}$ has ***enough injectives***.
2. If for all $X \in \mathcal{A}$, there is projective object P and epimorphism $X \to P$, then

we say $\mathcal{A}$ has ***enough projective***.

Definition 8.5

Let $\mathcal{A}$ be abelian category, then we say $\mathcal{A}$ is a ***Grothendieck category*** if:

1. $\mathcal{A}$ is cocomplete
2. $\mathcal{A}$ has generators
3. for all filtered small category I, the functor $\varinjlim : \mathcal{A}^I \to \mathcal{A}$ is exact, i.e. for any arrow $\alpha \to \beta \to \gamma$ in $\mathcal{A}^I := \mathrm{Fun}(I, \mathcal{A})$, we have

$$\forall i \in I, 0 \to \alpha(i) \to \beta(i) \to \gamma(i) \to 0 \text{ is eaxct}$$

implies

$$0 \to \varinjlim \alpha \to \varinjlim \beta \to \varinjlim \gamma \to 0 \text{ is exact}$$

As an immediate application we see in a Grothendieck category, for any small set I, the functor $\bigoplus_I : \mathcal{A}^I \to \mathcal{A}$ is exact.

Theorem 8.6: Grothendieck

Let $\mathcal{A}$ be Grothendieck category. Let $\mathcal{I}$ be the full subcategory of all injective objects, let the inclusion be $\iota : \mathcal{I} \to \mathcal{A}$. Then there is a functor $F : \mathcal{A} \to \mathcal{I}$ together with $\phi : \mathrm{Id}_{\mathcal{A}} \to \iota F$ so that

$$\phi_X : X \to F(X)$$

is monomorphism. Specifically, $\mathcal{A}$ has enough injectives.

Corollary 8.6.1

Every Grothendieck category $\mathcal{A}$ has injective cogenerator.

Now we move to ind and pro construction.

Definition 8.7

Let $\mathcal{C}$ be a category.

- If $X \in \mathrm{Obj}(\mathcal{C}^\wedge)$ can be represented as "$\varinjlim$"X_i, where the index is a filtered small category, and $X_i \in \mathcal{C}$, then we say X is an ***ind-object*** in $\mathcal{C}$.
- If $X \in \mathrm{Obj}(\mathcal{C}^\vee)$ can be represented as "$\varprojlim$"X_i, where the index is a filtered small category, and $X_i \in \mathcal{C}$, then we say X is an ***pro-object*** in $\mathcal{C}$.

The collection of all ind-objects of $\mathcal{C}$ forms the category $\mathrm{Ind}\,\mathcal{C}$, and the collection of all pro-objects form the category $\mathrm{Pro}\mathcal{C}$. Clearly we have fully faithful functor $\mathcal{C} \to \mathrm{Ind}\,\mathcal{C}$ and $\mathcal{C} \to \mathrm{Pro}\mathcal{C}$. Also, by definition of $\mathcal{C}^\wedge$ and $\mathcal{C}^\vee$, we see $(\mathrm{Ind}\,\mathcal{C})^{\mathrm{opp}} \cong \mathrm{Pro}(\mathcal{C}^{\mathrm{opp}})$. From now on we will write ind and pro objects by "$\varinjlim$"X_i and "$\varprojlim$"X_i.

Lemma 8.8

Let $X = \text{“}\varinjlim\text{”}X_i$ *and* $Y = \text{“}\varinjlim\text{”}Y_j$ *be ind-objects, then there is canonical bijection*

$$\mathrm{Hom}_{\mathrm{Ind}\,\mathcal{C}}(X,Y) \cong \varprojlim_i \varinjlim_J \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j)$$

Similarly for pro-objects X, Y *we have*

$$\mathrm{Hom}_{\mathrm{Pro}\,\mathcal{C}}(X,Y) \cong \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j)$$

Example 8.9

Let k be a field, and (**FinVect**) be the category of finite dimensional k-vector spaces. Then we will see

$$\mathcal{A} := \mathrm{Ind}(\mathbf{FinVect}) \cong (\mathbf{Vect}) := \mathcal{B}$$

where (**Vect**) is the category of k-vector spaces. Indeed, define functor $\mathcal{A} \to \mathcal{B}$ as follows: it maps “$\varinjlim$”V_i to $V := \varinjlim V_i$. To see this is fully faithful, observe

$$\mathrm{Hom}_k(V,W) \cong \varprojlim_i \varinjlim_j \mathrm{Hom}_k(V_i, W_j)$$

Indeed, to give a linear map $f : V \to W$ is the same as to give a family of compactible family of linear maps $f_i : V_i \to W$, while filtered $\varinjlim$ in (**Vect**) says each f_i will factor through some W_j.

This functor is also essentially surjective. Indeed, given vector space V, then the set of finite dimensional subspaces $V^\flat$ with inclusion relation gives a filtered poset, and clearly $\varinjlim V^\flat \cong V$.

Next, we will record two results that we will use, without proof:

Theorem 8.10

Let $\mathcal{C}$ *be abelian category, then* $\mathrm{Ind}\,\mathcal{C}$ *is also abelian, and* $\mathcal{C} \to \mathrm{Ind}\,\mathcal{C}$ *is fully faithful exact functor. Dually,* $\mathrm{Pro}\mathcal{C}$ *is abelian and* $\mathcal{C} \to \mathrm{Pro}\mathcal{C}$ *is fully faithful exact functor.*

Theorem 8.11

If $\mathcal{C}$ *is small abelian category, then* $\mathrm{Ind}\,\mathcal{C}$ *is Grothendieck category.*

Now we are ready to prove the Mitchell embedding theorem.

Lemma 8.12

Let $\mathcal{C}$ *be cocomplete abelian category with projective generators, and* $O \subseteq \mathrm{Obj}(\mathcal{C})$ *a set. Then there exists projective generator* S *such that all* $X \in O$ *can be represented* *as quotient object of S.*

Lemma 8.13

Let $S \in \mathrm{Obj}(\mathcal{C})$ be a projective generator of abelian category $\mathcal{C}$, and let $R = \mathrm{Hom}_{\mathcal{C}}(S,S)$. This gives fully faithful functor

$$G := \mathrm{Hom}_{\mathcal{C}}(S,\cdot) : \mathcal{C} \to (\textbf{\textit{Mod}-}R)$$

If $X \in \mathrm{Obj}(\mathcal{C})$ can be represented as the direct sum of finitely many copies of S, then the map

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{(\textbf{\textit{Mod}-}R)}(GX,GY)$$

induced by G is bijective for all $Y \in \mathcal{C}$.

Note R is a ring as we can add $f : S \to S$ and $g : S \to S$ since $\mathcal{C}$ is abelian, and we can multiply f and g by composition.

Proof. Take $m \geq 0$ and short exact sequence

$$0 \to X' \to S^{\oplus m} \to X \to 0$$

and apply G to it. Consider the diagram in (**AbGrp**)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(X,Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(S^{\oplus m},Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(X',Y) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_R(GX,GY) & \longrightarrow & \mathrm{Hom}_R(G(S^{\oplus m}),GY) & \longrightarrow & \mathrm{Hom}_R(GX',GY) \end{array} \tag{Eq. 8.1}$$

where all the vertical arrows are induced by fully faithful G, thus its monomorphism. Observe

$$\mathrm{Hom}_{\mathcal{C}}(S,Y) \xrightarrow{\mathrm{Id}} GY \xleftarrow[\psi \mapsto \psi(1_R)]{\sim} \mathrm{Hom}_R(R,GY) = \mathrm{Hom}_R(GS,GY)$$

where the composition is equal the homomorphism induced by G. Since G is additive, we see the middle vertical arrow in Eq. 8.1 is isomorphism. Then its not hard to see the left vertical arrow is also isomorphism.

Theorem 8.14: Mitchell Embedding

If $\mathcal{A}$ is small abelian category, then there is ring R and fully faithful exact functor $F : \mathcal{A} \to (\textbf{\textit{Mod}-}R)$.

Proof. Since $\mathcal{A}^{\mathrm{opp}}$ is still small abelian, Theorem 8.11 says $\mathrm{Ind}(\mathcal{A}^{\mathrm{opp}})$ is Grothendieck category, and hence it has an injective cogenerator, by Corollary 8.6.1. Then

$$\mathrm{Pro}(\mathcal{A}) \cong \mathrm{Ind}(\mathcal{A}^{\mathrm{opp}})^{\mathrm{opp}}$$

and so we see $\text{Pro}(\mathcal{A})$ has a projective generator. Also, we see $\mathcal{A} \to \text{Pro}\mathcal{A}$ is fully faithful exact functor between abelian categories, and so we can view $\mathcal{A}$ as a full subcategory of $\text{Pro}\mathcal{A}$.

Now apply Lemma 8.12 with $O = \mathcal{A}$, we get a projective generator S of $\text{Pro}(\mathcal{A})$, where all objects of $\mathcal{A}$ can be represented as quotients of S. Now consider the functor

$$\mathcal{A} \to \text{Pro}(\mathcal{A}) \xrightarrow{G} (\textbf{Mod-}R)$$

$$G := \text{Hom}_{\text{Pro}(\mathcal{A})}(S, \cdot), \quad R := \text{Hom}_{\text{Pro}\mathcal{C}}(S)$$

and denote the composition to be F. Then since $\mathcal{A} \to \text{Pro}\mathcal{A}$ and $\text{Pro}\mathcal{A} \xrightarrow{G} (\textbf{Mod-}R)$ are all exact, we see F is exact. By Lemma 8.13 we know G is fully faithful, thus F is fully faithful, thus we are done.

9 Injective Modules, Revisit

Well, we investigated the injective and projective objects in $(R\textbf{-Mod})$, along with flat modules. Namely, R-module I is injective if $\text{Hom}(-, I)_1$ is exact (here $\text{Hom}(-, I)_1$ is the contravariant hom functor from $(R\textbf{-Mod})$ to $(R\textbf{-Mod})^{\text{opp}}$), projective if $\text{Hom}(I, -)$ is exact, and flat if $\otimes I$ is exact.

Proposition 9.1

Consider short exact sequence $0 \to I \xrightarrow{f} M \xrightarrow{g} P \to 0$. *If I is injective or P projective, then this short exact sequence splits.*

Proposition 9.2: Baer Criterion

Module I is injective if and only if for any left ideal $\mathfrak{a}$ in R, any R-module morphism $f : \mathfrak{a} \to I$ can be extended to R-module morphism $R \to I$.

Proof. ($\Rightarrow$): Injectivity says for any $g : M' \to M$, every $h : M' \to I$ can be extended to $\tilde{h} : M \to I$. Now take $M' = \mathfrak{a}$ and $M = R$, and the arrow $M' \to M$ be inclusion.

($\Leftarrow$): Consider poset $(\mathcal{P}, \leq)$ where $\mathcal{P}$ consists of (M_1, h_1), where M_1 satisfies $M' \subseteq M_1 \subseteq M$, and $h_1 : M_1 \to I$ is an extension of h. We define $(M_1, h_1) \leq (M_2, h_2)$ iff $(M_1 \subseteq M_2)$ and $(h_2|_{M_1} = h_1)$. By standard application of Zorn's lemma, we see $\mathcal{P}$ has a maximal element. We claim if (M_1, h_1) is maximal then we must have $M_1 = M$.

Suppose $M_1 \neq M$, then take $x \in M \backslash M_1$, and define $\mathfrak{a} = \{r \in R : rx \in M_1\}$. Then we get

$$\phi : \mathfrak{a} \xrightarrow{r \mapsto rx} M_1 \xrightarrow{h_1} I$$

By assumption we can extend ϕ to $\tilde{\phi} : R \to I$. Now let $h_2 : M_2 \to I$ where $M_2 = M_1 + Rx$ and

$$h_2(x_1 + rx) = h_1(x_1) + \tilde{\phi}(r)$$

By definition of ϕ we see h_2 is well-defined, and thus $(M_2, h_2) > (M_1, h_1)$, a contradiction. Thus we are done.

Now let us study injective and projective objects in $\mathbb{Z}$-modules.

Definition 9.3

Let A be $\mathbb{Z}$-module. Then A is divisible if $n \neq 0$ implies $nA = A$.

Lemma 9.4

An $\mathbb{Z}$-module I is divisible if and only if I is injective $\mathbb{Z}$-module.

Proof. Let $n \neq 0$, then we see we get

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, I) & \xrightarrow{\text{restriction}} & \mathrm{Hom}_{\mathbb{Z}}(n\mathbb{Z}, I) \\ \cong \downarrow \phi \mapsto \phi(1) & & \cong \downarrow \phi \mapsto \phi(n) \\ I & \xrightarrow{a \mapsto na} & I \end{array}$$

is commutative. Because non-zero ideals of $\mathbb{Z}$ are given by $n\mathbb{Z}$, now just apply Proposition 9.2.

Lemma 9.5

Every $\mathbb{Z}$-module A can be embedded into injective $\mathbb{Z}$-module.

Proof. Represent A as $\mathbb{Z}^{\oplus X}/N$ for some index set X. Now consider embedding

$$\mathbb{Z}^{\oplus X}/N \to \mathbb{Q}^{\oplus X}/N$$

and note $\mathbb{Q}^{\oplus X}/N$ is divisible.

Corollary 9.5.1

*The category (**AbGrp**) has enough injectives.*

Now recall given ring R and S, left S-module N and (S,R)-bimodule P, we can give $\mathrm{Hom}_{(S\textbf{-Mod})}(P,N)$ a left R-module structure by

$$p(rf) = (pr)f, \quad p \in P, r \in R, f : P \to N$$

Lemma 9.6

Let R,S be rings, $N \in (S\textbf{-Mod})$, $P \in (S\textbf{-Mod-}R)$. If N is injective S-module, P is flat right R-module (i.e. right R-module), then $\mathrm{Hom}_{(S\textbf{-Mod})}(P,N)$ is injective left R-module.

Proof. Ntoe we have natural isomorphism

$$\mathrm{Hom}_{(R\textbf{-Mod})}(-, \mathrm{Hom}_{(S\textbf{-Mod})}(P,N)) \xrightarrow{\sim} \mathrm{Hom}_{(S\textbf{-Mod})}(P \otimes_R -, N)$$

where we used adjunction of hom and tensor. But the assumption says the functor on the right is exact.

Theorem 9.7

Every R-module M can be embedded into injective R-module.

Proof. View M as left $\mathbb{Z}$-module embedded into injective left $\mathbb{Z}$-module J. View R as $(\mathbb{Z},R)$-bimodule. Then we see $\mathrm{Hom}_{(\mathbb{Z}\textbf{-Mod})}(R,J)$ is injective by Lemma 9.6. Thus

$$M \xrightarrow[x \mapsto (r \mapsto rx)]{\sim} \mathrm{Hom}_{(R\textbf{-Mod})}(R,M) \to \mathrm{Hom}_{(\mathbb{Z}\textbf{-Mod})}(R,M) \to \mathrm{Hom}_{(\mathbb{Z}\textbf{-Mod})}(R,J)$$

are all embedding of R-modules.

Corollary 9.7.1

Let R be a ring, then $(R\textbf{-Mod})$ has enough injectives.

From the above, we can actually extract the following lemma.

Lemma 9.8: Injective Production Lemma

Let R be commutative ring, S an R-algebra. If I is an injective R-module, then $\mathrm{Hom}_R(S,I)$ is an injective S-module.

Remark 9.9

We note direct product of injective modules is still injective, but direct sum of injective modules need not be injective.

To end this section, we consider a minimal way to embed M into injective I. This $E(M)$ will be called injective envelope or injective hull.

Definition 9.10

Let $M \subseteq E$ be R-modules. We say M is ***essential submodule*** if $E \cap E_0 \neq 0$ for all submodules $E_0 \neq (0)$ in E.

For example, $\mathbb{Z}$ is essential in $\mathbb{Q}$.

Proposition 9.11

1. *If $M \subseteq E$ and E is injective, then we can find a minimal $F \subseteq E$ containing M such that $M \subseteq F$ is essential.*
2. *The F above will be injective.*
3. *If we pick another such F', then we get $F \cong F'$.*

Example 9.12

Let $M = \mathbb{Z}/p\mathbb{Z}$ for some prime p, then $E(M) = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.

10 Complexes

A huge portion of this and next few section are from a book by Jean Gallier in 2022 (homology, cohomology, and sheaf cohomolog

We begin with a little bit motivation on why we need cohomology/homology.

We know $\mathrm{Hom}(-, M)$ is not exact. We want to measure how far it is away from being exact, and the way we do it is by (co)homology. Let us see this in action.

Let Γ be a group, we say M is Γ-module if it is an abelian group together with an action of Γ from the left. Then, we get a functor $I : (\Gamma\textbf{-Mod}) \to (\textbf{AbGrp})$ by sending

$$M \mapsto M^\Gamma := \{m \in M : \gamma m = m, \forall \gamma \in \Gamma\}$$

This functor is not left exact, i.e. given

$$0 \to M' \to M \to M'' \to 0 \qquad \text{(Eq. 10.1)}$$

then we only get $0 \to (M')^\Gamma \to M^\Gamma \to (M'')^\Gamma$.

Now we look at why we fail to get surjection at the end. Well, if we pick $m'' \in (M'')^\Gamma$, then we want to know why this is not in the image of M^Γ. Since we have Eq.

10.1, we know for sure we can find $m \in M$ that maps to m''. The problem is we cannot ascertain this m is Γ-invariant, i.e. it may not be in M^Γ.

However, what we know is that, given such m, for all $\gamma \in \Gamma$, the difference

$$m'_\gamma := m - \gamma m \in M'$$

when we view M' as a subset of M. In other word, we get an assignment

$$\begin{aligned} \Gamma &\to M \\ \gamma &\mapsto m'_\gamma := m - \gamma m \end{aligned}$$

that satisfies $m'_{\gamma_1} + \gamma_1 m'_{\gamma_2} = m'_{\gamma_1\gamma_2}$. Call such a map $\Gamma \to M'$ satisfying this relation a 1-cocycle. Then the set of all 1-cocycles forms an abelian group, and we denote this by $Z^1(\Gamma, M')$.

The key observation is that our element $m \in M$ lies in M^Γ if and only if the cocycle $m'_\gamma = m - \gamma m = 0$. In particular, if $Z^1(\Gamma, M')$ is zero, then we know our functor $M \mapsto M^\Gamma$ is exact at the short exact sequence Eq. 10.1. Equivalently, $Z^1(\Gamma, M')$ measures how far away is I from left exact.

Next, we note there might be more than one element in M that maps to m''. However, all such choices will be related by an element m' of M'. That is, m_1 and m_2 both maps to m'' iff $m_1 = m_2 + m'$ for some $m' \in M'$. In other word, to know how bad the functor $M \mapsto M^\Gamma$ is not exact (on the left), we want to quotient $Z^1(\Gamma, M')$ by all the 1-cocycles $\gamma \mapsto b_\gamma$ for which we can find $m' \in M'$ so $b_\gamma = m' - \gamma m'$. Denote this subgroup by $B^1(\Gamma, M')$

Thus, at this point, we see the element $m'' \in (M'')^\Gamma$ defines an element in $Z^1(\Gamma, M')$, which is well-defined up to $B^1(\Gamma, M')$, i.e. we get a map

$$M'' \to H^1(\Gamma, M') := Z^1(\Gamma, M')/B^1(\Gamma, M')$$

that fits into the exact sequence $0 \to (M')^\Gamma \to M^\Gamma \to (M'')^\Gamma$, and $H^1(\Gamma, M')$ measures whether this short exact sequence is exact or not.

Now let us move to general theory. Note by Mitchell embedding, we can assume the abelian category we are working with will be some R-modules. The convention we are using will be that chain complexes have indices at bottom, and maps going down, while cochain complexes have indices at top, and maps going up.

Definition 10.1

Let $\mathcal{A}$ be an abelian category. We define the category $(\mathbf{Chain})(\mathcal{A})$ as follows:

1. objects of $(\mathbf{Chain})(\mathcal{A})$ will be a sequence $(C_n, d_n)_{n\in\mathbb{Z}}$, where $C_n \in \mathcal{A}$ and $d_n : C_n \to C_{n-1}$, such that $d_{n+1} \circ d_n = 0$ for all n. The map d_n is called the ***nth differential***, and an element of $(\mathbf{Chain})(\mathcal{A})$ is called a ***chain complex***.
2. morphisms from (C_n, d_n) to (B_n, b_m) are given by $(f_n)_{n\in\mathbb{Z}}$ so that it makes

the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \longrightarrow & B_{n+1} & \xrightarrow[b_{n+1}]{} & B_n & \xrightarrow[b_n]{} & B_{n-1} & \longrightarrow & \cdots \end{array}$$

The category of cochain complexes in $\mathcal{A}$ is defined to be the category of chain complexes of $\mathcal{A}^{\text{opp}}$.

Definition 10.2

Given a chain complex $C_\bullet = (C_n, d_n)$ in $\mathcal{A}$, we define the ***ith homology object*** to be

$$H_i(C_\bullet) := Z_i(C_\bullet)/B_i(C_\bullet), \quad \text{where } Z_i(C_\bullet) := \ker(d_n), \quad B_i(C_\bullet) = \operatorname{im}(d_{n+1})$$

Similarly, we define the ith cohomology object of cochain complex $(C^n, d^n)_{n\in\mathbb{Z}}$ to be

$$H^i(C^\bullet) := \ker(d^n)/\operatorname{im}(d^{n-1})$$

where we also use the notation $Z^n(C^\bullet) := \ker(d^n)$ and $B^n(C^\bullet) := \operatorname{im}(d^{n-1})$.

Remark 10.3

We will call the elements of $Z_n(A_\bullet)$ as n-cycles, and elements of $B_n(A_\bullet)$ as n-boundaries. Similarly elements of $Z^n(A^\bullet)$ are called n-cocycles, and elements of $B^n(A^\bullet)$ as n-coboundaries.

We said $(\mathbf{Chain})(\mathcal{A})$ is a category, but we will not check it. It is actually an abelian category, we will also not check it. For example, its not hard to see $C_\bullet \oplus B_\bullet = (C_n \oplus B_n, d_n \oplus b_n)$ and so on.

Definition 10.4

A chain complex is ***bounded above*** if $C_n = 0$ for $n \gg 0$, and ***bounded below*** if $C_n = 0$ for $n \ll 0$. It is called ***bounded*** if it is both bounded above and bounded below.

The situation for cochain complexes is dual to the case of chain complexes. We will use $(\mathbf{Chain})^b(\mathcal{A})$ to denote the bounded chain complexes, which forms an abelian subcategory of $(\mathbf{Chain})(\mathcal{A})$.

Now we note the assignment H_i is in fact a functor. Indeed, let $f : A_\bullet \to B_\bullet$, then f induces a map $H_i(A_\bullet) \to H_i(B_\bullet)$. Indeed, take $x \in \ker(a_n)$, then it maps to 0 by $a_n : A_n \to A_{n-1}$. On the other hand, since f is a morphism of chains, we see $f_n(x)$ must be mapped to 0 by b_n. In other word, $f_n : \ker(a_n) \to \ker(b_n)$, and hence we see $\ker(a_n)/\operatorname{im}(a_{n+1}) \to \ker(b_n)/\operatorname{im}(b_{n+1})$ defined by $[x] \mapsto [f_n(x)]$ is well-defined. We denote this map by $H_i(f) : H_i(A_\bullet) \to H_i(B_\bullet)$.

Now let us work out snake lemma.

Lemma 10.5: Snake Lemma

*In (R-**Mod**), suppose we have*

$$\begin{array}{ccccccc} & & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d''} & & \\ 0 & \longrightarrow & N' & \xrightarrow[a]{} & N & \xrightarrow[b]{} & N'' & & \end{array}$$

with all the rows exact. Then, we have an exact sequence

$$\begin{array}{ccccc} \ker(d') & \longrightarrow & \ker(d) & \longrightarrow & \ker(d'') \\ & & \xrightarrow{\delta} & & \\ \operatorname{coker}(d') & \longrightarrow & \operatorname{coker}(d) & \longrightarrow & \operatorname{coker}(d'') \end{array}$$

Proof. We have

$$\begin{array}{ccccccccc} & & \ker(d') & & \ker(d) & & \ker(d'') & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \operatorname{coker}(d') & & \operatorname{coker}(d) & & \operatorname{coker}(d'') & & \end{array}$$

We need to do a diagram chase. The exactness for the ker and coker are easy, so we will not do it in detail. To define $\ker(d') \to \ker(d)$, take $m' \in \ker(d')$, which means $d'(m') = 0 \in N'$, and thus $a(d'(m')) = 0 \in N$, and thus we see $d(f(m')) = a(d'(m')) = 0$, i.e. $f(m') \in \ker(d)$. Hence, $\ker(d') \to \ker(d)$ is given by $m' \mapsto f(m')$. The others are similar and we wouldn't check the exactness.

Construction 10.6

Here we define δ. Let $m'' \in \ker(d'') \subseteq M''$, then since $g : M \to M''$ is surjective, we can find $m \in M$ so that $g(m) = m''$. However, note $d''(m'') = 0$ by assumption, thus by commutativity we must have $b(d(m)) = 0 = d''(g(m))$. In other word, we see $d(m) \in \ker(b) \subseteq N$. However, $\ker(b) = \operatorname{im}(a)$, thus we can find $n' \in N'$ so that $a(n') = d(m)$. We define $\delta(m'') := n' + \operatorname{im}(d') \in N'/\operatorname{im}(d') = \operatorname{coker}(d')$.

We need to show this map δ is well-defined. To that end, if you chase through the definition, the first place it may cause ambiguity is when we pick $m \in M$ such that $g(m) = m''$.

Hence, suppose $g(m_1) = g(m_2) = m''$. But then this means $g(m_1 - m_2) = 0$, i.e. $m_1 - m_2 \in \ker(g) = \operatorname{im}(f)$, hence we get $m'_0 \in M'$ so that $f(m'_0) = m_1 - m_2$. But then note $a(d'(m'_0)) = d(f(m'_0)) = d(m_1 - m_2) = d(m_1) - d(m_2) \in \operatorname{im}(a)$. Now, by the above, we can pick $n'_1, n'_2 \in N'$ so $a(n'_1) = d(m_1)$ and $a(n'_2) = d(m_2)$ as above, and thus we need to show $n'_1 + \operatorname{im}(d') = n'_2 + \operatorname{im}(d')$, i.e. $n'_1 - n'_2 \in \operatorname{im}(d')$. But $d(m_1) - d(m_2) = a(n'_1 - n'_2) = a(d'(m'_0))$. Now observe a is injective, thus we must have $n'_1 - n'_2 = d'(m'_0) \in \operatorname{im}(d')$. This shows our map δ is well-defined at the choice of $m \in M$ such that $g(m) = m''$.

Next, we also made a choice of n' such that $a(n') = d(m)$. Suppose we have $a(n'_1) = a(n'_2) = d(m)$, but then since a is injective we must have $n'_1 = n'_2$, and so there is no ambiguity there.

This concludes δ is well-defined. It remains to show we have exact sequence at δ.

The first thing we need to show is $\operatorname{im}(\ker(d) \to \ker(d'')) = \ker(\delta)$. Well,

$$\operatorname{im}(\ker(d) \to \ker(d''))$$

by definition is given by some $m'' \in \ker(m'')$ such that we can find $m \in \ker(d)$ so $g(m) = m''$. But if you chase through the definition of δ, we see $d(m) = 0$ means $\delta(m'') = 0$, hence $\operatorname{im}(\ker(d) \to \ker(d'')) \subseteq \ker(\delta)$. Next, suppose $m'' \in \ker(\delta)$, i.e. $m'' \in \ker(d'')$ and $\delta(m'') = 0$. Inherit the notation of 10.6, we see this means $n' \in \operatorname{im}(d')$, i.e. we can find $m' \in M'$ so $d'(m') = n'$. However, note since all the squares must commute, we have $d(f(m')) = a(d'(m'))$, where we know $a(d'(m')) = d(m)$. Thus, we see $d(f(m') - m) = 0$, i.e. $f(m') - m \in \ker(d)$. In particular, observe $g(f(m') - m) = g(f(m')) - g(m) = m''$, but $g(f(m')) = 0$ because all the rows are exact by assumption. In other word, we see $m'' \in \operatorname{im}(\ker(d) \to \ker(d''))$, as it is the image of $f(m') - m$ under the map $\ker(d) \to \ker(d'')$.

To conclude the proof, it remains to show $\operatorname{im}(\delta) = \ker(\operatorname{coker}(d') \to \operatorname{coker}(d))$. This is done by the following.

Exercise

Show $\operatorname{im}(\delta) = \ker(\operatorname{coker}(d') \to \operatorname{coker}(d))$

We will use this result to show that given a short exact sequence of chains $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$, then we get a long exact sequence of homology objects. Before that, we need the following lemma.

Lemma 10.7

Let $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$ *be a short exact sequence of chain complexes in* (R-**Mod**). *Then we have exact sequence*

$$H_n(A_\bullet) \xrightarrow{H_n(f)} H_n(B_\bullet) \xrightarrow{H_n(g)} H_n(C_\bullet)$$

for all $n \in \mathbb{Z}$.

Proof. We need to first define $H_n(f)$, but it is just given by

$$H_n(f)[a] = [f_n(a)]$$

We need to check this is well-defined. Suppose $a \in Z_n(A_\bullet) = \ker(a_n)$, then $a_n(a) = 0$, and thus $f_{n-1}(a_n(a)) = b_n(f_n(a)) \in B_{n-1}$, i.e. $f_n(a) \in \ker(b_n)$ as desired, i.e. we get $f_n : \ker(a_n) \to \ker(b_n)$. Now we need to show well-defined. For that, suppose $[b] = [a] \in H_n(A_\bullet)$, i.e. $a - b \in B_n(A_\bullet)$. However, this means we can find $y \in A_{n+1}$ so $a_{n+1}(y) = a - b$. But then

$$f_n(a-b) = f_n(a_{n+1}(y)) = b_{n+1}(f_{n+1}(y))$$

which shows $f_n(a) - f_n(b) \in \operatorname{im}(b_{n+1})$, i.e. $[f_n(a)] = [f_n(b)]$ in $H_n(B_\bullet)$. This shows $H_n(f)$ and $H_n(g)$ are well-defined, it remains to show they are exact.

For this purpose, let's write down the maps at hand:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_{n+1}} & & \downarrow{\scriptstyle b_{n+1}} & & \downarrow{\scriptstyle c_{n+1}} & & \\
0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_n} & & \downarrow{\scriptstyle b_n} & & \downarrow{\scriptstyle c_n} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \longrightarrow & 0
\end{array}$$

First we show $\operatorname{im}(H_n(f)) \subseteq \ker(H_n(g))$. Observe

$$H_n(g)(H_n(f)[x]) = H_n(g)([f_n(x)]) = [g_n(f_n(x))] = [0]$$

as $gf = 0$ by assumption.

Now it remains to show $\ker(H_n(g)) \subseteq \operatorname{im}(H_n(f))$. Suppose $H_n(g)[x] = 0$, i.e. $g_n(x) \in \operatorname{im}(c_{n+1})$. The goal is to find some $[z] \in H_n(A_\bullet)$ so $[f_n(z)] = [x]$.

Since $g_n(x) \in \operatorname{im}(c_{n+1})$, we can find $y \in C_{n+1}$ so $g_n(x) = c_{n+1}(y)$. Since g_{n+1} is surjective, we can find $x_1 \in B_{n+1}$ so $y = g_{n+1}(x_1)$, and thus we see

$$g_n(b_{n+1}(x_1)) = c_{n+1}(g_{n+1}(y)) = g_n(x)$$

Thus we see

$$g_n(b_{n+1}(x_1) - x) = 0$$

and hence $b_{n+1}(x_1) - x \in \ker(g_n) = \operatorname{im}(f_n)$, i.e. we can find $z \in A_n$ so $f_n(z) = b_{n+1}(x_1) - x$. We claim this z in fact lives in $\ker(a_n)$. To see this, it suffices to show $a_n(z) = 0$. However, since f is injective, it suffices to show $f_{n-1}(a_n(z)) = 0$. Now observe we have

$$f_{n-1}(a_n(z)) = b_n(f_n(z))$$

where

$$f_n(z) = b_{n+1}(x_1) - x \Rightarrow b_n(f_n(z)) = b_n(b_{n+1}(x_1)) - b_n(x)$$

where $b_n \circ b_{n-1} = 0$ and $b_n(x) = 0$ as $x \in \ker(b_n)$. Hence we see $z \in \ker(a_n)$ and so $[z] \in H_n(A_\bullet)$. Now we see

$$[f_n(z)] = [b_{n+1}(x_1) - x] = [x] \in \ker(b_n)/\operatorname{im}(b_{n+1})$$

Theorem 10.8: Zig-Zag Lemma

Let $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$ *be a short exact sequence of chain complexes in* $(R\text{-}\mathbf{Mod})$*. Then we have a long exact sequence*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_n(A_\bullet) & \xrightarrow{H_n(f)} & H_n(B_\bullet) & \xrightarrow{H_n(g)} & H_n(C_\bullet) & & \\ & & & & \delta & & & & \\ & & H_{n-1}(A_\bullet) & \xrightarrow[H_{n-1}(f)]{} & H_{n-1}(B_\bullet) & \xrightarrow[H_{n-1}(g)]{} & H_{n-1}(C_\bullet) & \longrightarrow & \cdots \end{array}$$

Proof. Consider the following sequence

$$\begin{array}{ccccccc} & & A_n/\operatorname{im}(a_{n+1}) & \xrightarrow{\overline{f}} & B_n/\operatorname{im}(b_{n+1}) & \xrightarrow{\overline{g}} & C_n/\operatorname{im}(c_{n+1}) & \longrightarrow & 0 \\ & & \downarrow \overline{a_n} & & \downarrow \overline{b_n} & & \downarrow \overline{c_n} & & \\ 0 & \longrightarrow & \ker(a_{n-1}) & \longrightarrow & \ker(b_{n-1}) & \longrightarrow & \ker(c_{n-1}) & & \end{array}$$

Here the vertical arrows are induced by the differentials. For example,

$$\overline{a_n} : A_n/\operatorname{im}(a_{n+1}) \to \ker(a_{n-1})$$

just sends $[x]$ to $a_n(x)$. This is well-defined because we know $\operatorname{im}(a_{n+1}) \subseteq \ker(a_n)$. In addition, we see $\overline{a_n}([x]) \in \ker(a_{n-1})$ because $a_{n-1}(\overline{a_n}[x]) = a_{n-1}(a_n(x)) = 0$. Thus we can apply snake lemma and obtain the following exact sequence

$$\begin{array}{ccccc} \ker(\overline{a_n}) & \longrightarrow & \ker(\overline{b_n}) & \longrightarrow & \ker(\overline{c_n}) \\ & & \delta_n & & \\ \operatorname{coker}(\overline{a_n}) & \longrightarrow & \operatorname{coker}(\overline{b_n}) & \longrightarrow & \operatorname{coker}(\overline{c_n}) \end{array}$$

However, we see

$$\ker(\overline{a_n}) = \ker(a_n)/\operatorname{im}(a_{n+1}) \quad \text{and} \quad \operatorname{coker}(\overline{a_n}) \cong \ker(a_{n-1})/\operatorname{im}(a_n)$$

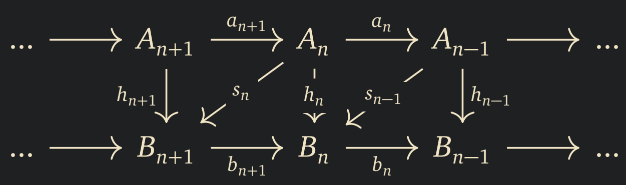
and so on. Indeed, note $\operatorname{coker}(\overline{a_n}) = \ker(a_{n-1})/\operatorname{im}(\overline{a_n})$, but $\operatorname{im}(\overline{a_n}) = \operatorname{im}(a_n)$ almost by definition. This concludes the proof, as it is not hard to see the vertical arrows are just $H_n(f)$ and $H_n(g)$ and $H_{n-1}(f)$ and $H_{n-1}(g)$.

Definition 10.9

Let $f, g : A_\bullet \to B_\bullet$, then a ***chain homotopy*** between f and g is a family $s = (s_n)_{n\in\mathbb{Z}}$ of morphisms $s_n : A_n \to B_{n+1}$ such that for all n we have

$$f_n - g_n = s_{n-1} \circ a_n + b_{n+1} \circ s_n$$

In terms of diagrams, a chain homotopy is given by a family of arrows as belows, where we set $h = f - g$:



so that every h_n is the sum of arrows on the side of the parallelogram.

Proposition 10.10

Let $f, g : A_\bullet \to B_\bullet$, and s is a chain homotopy between f and g. Then $H_n(f) = H_n(g)$ for all $n \in \mathbb{Z}$.

Proof. Let $[x] \in H_n(A_\bullet)$, with $x \in \ker(a_n)$. Then

$$(H_n(f) - H_n(g))[x] = [s_{n-1}(a_n(x)) - b_{n+1}(s_n(x))]$$

However, since $x \in \ker(a_n)$, we see we get

$$(H_n(f) - H_n(g))[x] = [s_{n-1}(0) - b_{n+1}(s_n(x))]$$

where $b_{n+1}(s_n(x)) \in \operatorname{im}(b_{n+1})$, i.e. $[b_{n+1}(s_n(x))] = [0]$. In other word, we get

$$H_n(f)[x] = H_n(g)[x]$$

for all $[x] \in H_n(A_\bullet)$.

Definition 10.11

A ***homotopy equivalence*** between chain complexes $C_\bullet$ and $D_\bullet$ consists of a pair (g, h) of chain maps $g : C \to D$ and $h : D \to C$ such that $h \circ g$ is chain homotopic to Id_C and $g \circ h$ is chain homotopic to Id_D.

11 Resolutions

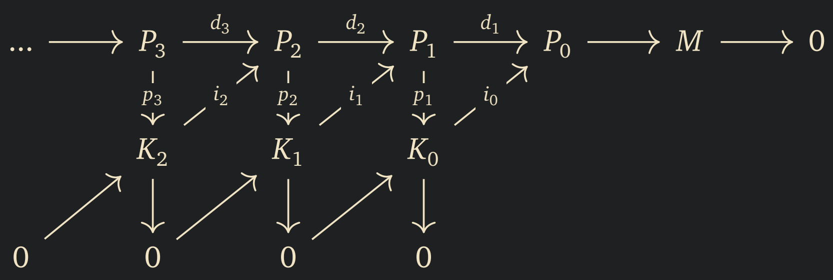
y for algebraic topology, algebraic geometry, and differential geometry).

Since (R-**Mod**) has enough injective and projective, let M be any module, then we can find projective P_0 and a surjection $p_0 : P_0 \to M$, such that $M \cong P_0 / \ker p_0$. However, $K_0 = \ker p_0$ is not necessarily projective, and thus we might as well take another projective module P_1 and surjection $p_1 : P_1 \to K_0$, with $K_0 \cong P_1 \cong \ker p_1$. However,

$K_1 = \ker p_1$ might not be projective, and so on. We can continue this indefinitely, where by induction, we get

$$0 \to K_n \xrightarrow{i_n} P_n \xrightarrow{p_n} K_{n-1} \to 0$$

with P_n projective, $K_n = \ker p_n$, and i_n the inclusion map for all $n \geq 1$. Overall, this sequence is illustrated by the following diagram



where $d_n : P_n \to P_{n-1}$ is defined by

$$d_n = i_{n-1} \circ p_n$$

for all $n \geq 1$.

Now observe $\ker d_n = \ker p_n = K_n$, and because p_n is surjective, we see

$$\operatorname{im} d_n = \operatorname{im} i_{n-1} = K_{n-1}$$

Therefore, $\operatorname{im} d_{n+1} = \ker d_n$ for all $n \geq 1$. At the end, we have $\operatorname{im} d_1 = K_0 = \ker p_0$ with p_0 surjective, and thus we conclude the above chain complex is exact.

Definition 11.1

Given any R-module M, a ***projective*** (resp. ***free***, resp. ***flat***) resolution is a chain complex $P_\bullet$ together with surjective $p_0 : P_0 \to M$, so that the following sequence

$$\ldots \to P_n \xrightarrow{d_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{p_0} M \to 0$$

is exact, and every P_n is projective (resp. free, resp. flat) module. We will often write the above exact chain complex as $P_\bullet \xrightarrow{p_0} M \to 0$.

Remark 11.2

An exact sequence of the above form where P_n are not necessarily projective (nor free, nor flat) is called a left acyclic resolution of M.

Since (R-**Mod**) also has enough injective, we can do the same thing for injective modules.

Definition 11.3

Given R-module M, a ***injective resolution*** is a cochain complex $I^\bullet$ together with injective $i_0 : M \to I^0$, so that the sequence

$$0 \to M \xrightarrow{i_0} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \ldots \to I^n \xrightarrow{d^n} I^{n+1} \to \ldots$$

is exact, and I^n is an injective module. We often write this as $0 \to M \xrightarrow{i_0} I^\bullet$.

Remark 11.4

An exact sequence of the above form where I^n are not necessarily injective is called a right acyclic resolution of M.

Proposition 11.5

Every R-module M admits a projective resolution and an injective resolution.

Now let $P_\bullet$ be a projective resolution of A, then we can apply the $\mathrm{Hom}(-,B)$ functor (which is left exact and contravariant) to this chain complex and get another chain complex

$$0 \to \mathrm{Hom}(P_0,B) \to \dots \to \mathrm{Hom}(P_{n-1},B) \to \mathrm{Hom}(P_n,B) \to \dots$$

Definition 11.6

For any two R-modules A and B, the ***n*th Ext group** $\mathrm{Ext}^n_R(A,B)$ is the nth cohomology groups of the cochain complex $\mathrm{Hom}(P_\bullet,B)$, where $P_\bullet$ is a projective resolution of A.

Now, note since $\mathrm{Hom}(-,B)$ is left exact, the exact sequence

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{p_0} A \to 0$$

gives exact sequence

$$0 \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(P_0,B) \to \mathrm{Hom}(P_1,B)$$

This implies

$$\mathrm{Hom}(A,B) \cong \ker(\mathrm{Hom}(d_1,B)) = H^0(\mathrm{Hom}(P_\bullet,B))$$

That is,

$$\mathrm{Ext}^0_R(A,B) \cong \mathrm{Hom}(A,B)$$

Example 11.7

If A itself is projective, then take $0 \to A \xrightarrow{\mathrm{Id}} A \to 0$, we see $\mathrm{Ext}^n_R(A,B) = 0$ for all $n \geq 1$.

On the other hand, if R is PID, then every module A admits a free resolution $0 \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{p_0} A \to 0$ and thus $\mathrm{Ext}^n_R(A,B) = 0$ for all $n \geq 2$.

Similar to $\mathrm{Hom}(-,B)$, for the covariant right exact functor $-\otimes_R B$, we can construct a chain complex $P_\bullet \otimes_R B$, i.e. we get

$$\dots \to P_n \otimes B \to P_{n-1} \otimes B \to \dots \to P_0 \otimes B \to 0$$

Definition 11.8

For any two R-modules A and B, the ***nth Tor groups*** $\mathrm{Tor}_R^n(A,B)$ is the nth homology group of the chain complex $P_\bullet \otimes B$, where $P_\bullet$ is a projective resolution of A.

Since $-\otimes B$ is right eaxct, we see $Tor_0^R(A,B) \cong A \otimes B$. Also, by the same argument, if R is PID, then $\mathrm{Tor}_0^R(A,B) = 0$ for $n \geq 2$.

By definition, we see $\mathrm{Tor}(A,B)$ and $\mathrm{Ext}(A,B)$ depends on the choice of projective resolution. However, as we will see later, any two resolutions $P_\bullet$ and $Q_\bullet$ give isomorphic Tor and Ext.

Before we move into prove this result, let us remark the following.

Observe the functor $\mathrm{Hom}(A,-)$ is covariant and left exact, thus for an injective resolution $I^\bullet$ of B, we get cochain complex $\mathrm{Hom}(A,I^\bullet)$, i.e.

$$\mathrm{Hom}(A,I^0) \to \mathrm{Hom}(A,I^1) \to \ldots \to \mathrm{Hom}(A,I^n) \to \mathrm{Hom}(A,I^{n+1}) \to \ldots$$

Definition 11.9

For any two R-modules A,B, the ***nth Ext' group*** $\mathrm{Ext}_R^{\prime n}(A,B)$ is the nth cohomology group of $\mathrm{Hom}(A,I^\bullet)$, where $I^\bullet$ is an injective resolution of B.

Since one can prove $\mathrm{Ext}_R^{\prime n}(A,B)$ is isomorphic to $\mathrm{Ext}_R^n(A,B)$, we will not distinguish those two and just call them the Ext group of A and B.

Our next task is to show the Ext's and Tor's are isomorphic when we choose two different projective resolutions. To be more precise, there is a chain homotopy equivalence between two projective resolutions $P_\bullet$ and $P'_\bullet$ of A (where a similar result holds for injective resolutions).

Theorem 11.10: Projective Comparison Theorem

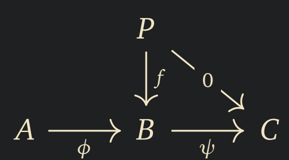
Let A,B be two R-modules. If:

1. *$P_\bullet \xrightarrow{\epsilon} A \to 0$ is a chain complex with all P_n projective, and*
2. *$X_\bullet \xrightarrow{\epsilon'} B \to 0$ is an exact sequence.*

Then any R-linear map $f : A \to B$ lifts to a morphism g from $P_\bullet$ to $X_\bullet$, and any two morphisms from $P_\bullet$ to $X_\bullet$ lifting f are chain homotopic.

What the last sentence means is that, if we have $h : P_\bullet \to X_\bullet$ which also satisfies $f \circ \epsilon = h_0 \circ \epsilon'$, then we must have g in the above theorem and h being chain homotopic.

Also, before we start to prove this result, let us just remark a general lifting procedure. Suppose we have the following commutative diagram



with bottom row exact and P projective, then we can find $s : P \to A$ which makes the whole diagram commute. To see this, just note $\psi \circ f = 0$ implies $\operatorname{im}(f) \subseteq \ker\psi = \operatorname{im}\phi$, and thus $\operatorname{im}(f) \subseteq \operatorname{im}(\phi)$, and so we can replace B by the image of ϕ, and get

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle f} & & \\ A & \xrightarrow[\phi]{} & \operatorname{im}(\phi) & \xrightarrow[\psi]{} & 0 \end{array}$$

where we can apply the definition of projective modules.

Proof. We prove the existence of the lift, stepwise, by induction. Since we have $\epsilon : P_0 \to A$ and $f : A \to B$, we get diagram

$$\begin{array}{ccccc} & & P_0 & & \\ & & \downarrow{\scriptstyle f\circ\epsilon} & & \\ X_0 & \xrightarrow{\epsilon'} & B & \longrightarrow & 0 \end{array}$$

As P_0 is projective, the map $g_0 : P_0 \to X_0$ exists and make the above diagram commutes. Assume the lift exists up to level n. We have the diagram

$$\begin{array}{ccccccc} P_{n+1} & \xrightarrow{d^P_{n+1}} & P_n & \xrightarrow{d^P_n} & P_{n-1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle g_{n-1}} & & \\ X_{n+1} & \xrightarrow{d^X_{n+1}} & X_n & \xrightarrow{d^X_n} & X_{n-1} & \longrightarrow & \dots \end{array} \qquad \text{(Eq. 11.1)}$$

so we get map $g_n \circ d_{n+1^P} : P_{n+1} \to X_n$ and a diagram

$$\begin{array}{ccccc} & & P_{n+1} & & \\ & & \downarrow{\scriptstyle g_n\circ d^P_{n+1}} & & \\ X_{n+1} & \xrightarrow[d^X_{n+1}]{} & X_n & \xrightarrow[d^X_n]{} & X_{n-1} \end{array}$$

where the bottom row is exact and $d^X_n \circ (g_n \circ d^P_{n+1}) = 0$. Indeed, by commutativity of Eq. 11.1, we get

$$d^X_n \circ g_n \circ d^P_{n+1} = g_{n-1} \circ d^P_n \circ d^P_{n+1} = 0$$

Now replace X_n by the image of $g_n \circ d^P_{n+1}$, then X_{n-1} becomes 0, and we can apply the fact P_{n+1} is projective and get a lifting $g_{n+1} : P_{n+1} \to X_{n+1}$, which makes the diagram

$$\begin{array}{ccccc} & \swarrow{\scriptstyle g_{n+1}} & P_{n+1} & & \\ & & \downarrow{\scriptstyle g_n\circ d^P_{n+1}} & & \\ X_{n+1} & \xrightarrow[d^X_{n+1}]{} & X_n & \xrightarrow[d^X_n]{} & X_{n-1} \end{array}$$

commutes.

It remains to show that if we have two liftings g and h, then they are chain homotopic. We will do this by induction.

For the base case, we have diagram

$$\begin{array}{ccccccc} & & P_0 & \xrightarrow{\epsilon} & A & \longrightarrow & 0 \\ & \overset{\exists s_0}{\swarrow} & {\scriptstyle g_0}\downarrow\downarrow{\scriptstyle h_0} & & \downarrow & & \\ X_1 & \xrightarrow[d_1^X]{} & X_0 & \xrightarrow{\epsilon'} & B & \longrightarrow & 0 \end{array}$$

As $\epsilon'(g_0 - h_0) = (f - f)\epsilon = 0$, the lower row is exact and P_0 is projective, thus we get $s_0 : P_0 \to X_1$ with $g_0 - h_0 = d_1^X \circ s_0$.

Assume, for the induction step, we have $s_0, \ldots, s_{n-1}$. Write $\Delta_n = g_n - h_n$, then we get the diagram

$$\begin{array}{ccccccccc} & & P_n & \xrightarrow{d_n^P} & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \ldots \\ & & {\scriptstyle \Delta_n}\downarrow & \overset{s_{n-1}}{\swarrow} & \downarrow{\scriptstyle \Delta_{n-1}} & & \downarrow{\scriptstyle \Delta_{n-2}} & & \\ X_{n+1} & \longrightarrow & X_n & \xrightarrow[d_n^X]{} & X_{n-1} & \longrightarrow & X_{n-2} & \longrightarrow & \ldots \end{array} \tag{Eq. 11.2}$$

This gives a map $\Delta_n - s_{n-1} \circ d_n^P : P_n \to X_n$ and a diagram

$$\begin{array}{ccccc} & & P_n & & \\ & & \downarrow{\scriptstyle \Delta_n - s_{n-1} \circ d_n^P} & & \\ X_{n+1} & \xrightarrow[d_{n+1}^X]{} & X_n & \xrightarrow[d_n^X]{} & X_{n-1} \end{array}$$

the bottom row is exact. If we can show $d_n^X \circ (\Delta_n - s_{n-1} \circ d_n^P) = 0$, then we can get a lift $s_n : P_n \to X_{n+1}$ making the diagram

$$\begin{array}{ccccc} & & P_n & & \\ & \overset{s_n}{\swarrow} & \downarrow{\scriptstyle \Delta_n - s_{n-1} \circ d_n^P} & & \\ X_{n+1} & \xrightarrow[d_{n+1}^X]{} & X_n & \xrightarrow[d_n^X]{} & X_{n-1} \end{array}$$

commute and we are done. By commutativity of Eq. 11.2, we see $d_n^X \circ \Delta_n = \Delta_{n-1} \circ d_n^P$ and so

$$d_n^X(\Delta_n - s_{n-1} \circ d_n^P) = \Delta_{n-1} \circ d_n^P - d_n^X \circ s_{n-1} \circ d_n^P$$

By induction hypothesis, we have

$$\Delta_{n-1} = g_{n-1} - h_{n-1} = s_{n-2} \circ d_{n-1}^P + d_n^X \circ s_{n-1}$$

and therefore

$$\begin{aligned} & \Delta_{n-1} \circ d_n^P - d_n^X \circ s_{n-1} \circ d_n^P \\ & = s_{n-2} \circ d_{n-1}^P \circ d_n^P + d_n^X \circ s_{n-1} \circ d_n^P - d_n^X \circ s_{n-1} \circ d_n^P \\ & = 0 \end{aligned}$$

This concludes the existence of s_n and we are done.

As an immediate result of this theorem, we see that:

Corollary 11.10.1

Given any R-linear map $f : A \to B$, if $P_\bullet$ and $P'_\bullet$ are projective resolutions of A and B respectively. Then f has a lift g from $P_\bullet$ to $P'_\bullet$. Furthermore, any two lifts of f are chain homotopic.

Theorem 11.11

Given any R-modules A, if $P_\bullet \xrightarrow{\epsilon} A \to 0$ and $P'_\bullet \xrightarrow{\epsilon'} A \to 0$ are two projective resolutions of A. Then $P_\bullet$ and $P_\bullet$ are homotopy equivalent.

Proof. By Proposition 11.10.1, the identity map $\mathrm{Id}_A : A \to A$ has a lift g from $P_\bullet$ to $P'_\bullet$, and a lift h from $P'_\bullet$ to $P_\bullet$. Thus $h \circ g$ is a lift of Id_A from $P_\bullet$ to $P_\bullet$, and since the identity map $\mathrm{Id}_{P_\bullet}$ is also a lift of Id_A, by Proposition 11.10.1 we get a chain homotopy from $h \circ h$ to $\mathrm{Id}_{P_\bullet}$. A similar argument shows $g \circ h$ is chain homotopic to $\mathrm{Id}_{P'_\bullet}$.

A dual result holds for injective resolutions, and all the proofs are similar. All we need to note is the following: if we have commutative

$$\begin{array}{ccccc} A & \xrightarrow{\psi} & B & \xrightarrow{\phi} & C \\ & \overset{0}{\searrow} & \downarrow{\scriptstyle f} & & \\ & & I & & \end{array}$$

with I injective, the upper sequence exact, then there is a map $\theta : C \to I$ which lifts f. In this case, just note $f \circ \psi = 0$, so $\operatorname{im}\psi \subseteq \ker f$, but $\ker\phi = \operatorname{im}\psi \subseteq \operatorname{im} f$ and so $\ker\phi \subseteq \ker f$, i.e. we get unique map $\overline{f} : B/\ker\phi \to I$, and now we back to the definition of injective module to get our desired θ.

We record the result here, as well as its implication.

Theorem 11.12: Injective Comparison Theorem

Given any R-linear map $f : A \to B$ in $(R\text{-}\mathbf{Mod})$, injective resolutions $0 \to A \to I^\bullet$, $0 \to B \to I'^\bullet$. Then f has a lift g from $I^\bullet$ to $I'^\bullet$. Furthermore, any two lifts of f are chain homotopic.

From this, we get dual results to Theorem 11.10, and in particular the following result:

Theorem 11.13

Given any R-modules A, if $0 \to A \xrightarrow{\epsilon} I^\bullet$ and $0 \to A \xrightarrow{\epsilon'} I'^\bullet$ are two injective resolutions. Then $I^\bullet$ and $I'^\bullet$ are homotopy equivalent.

Now we are move towards the definition of derived functors, but before that, we need some results about exact sequences of chain complexes.

Proposition 11.14: Projective Horseshoe Lemma

Let $\mathcal{A}$ be an abelian category, and exact short sequence $0 \to A' \xrightarrow{\phi} A \xrightarrow{\psi} A'' \to 0$. Suppose we have projective resolution $\mathbf{P}' := P'_\bullet \xrightarrow{\epsilon'} A' \to 0$ and $\mathbf{P}'' : P''_\bullet(A)P''_\bullet \xrightarrow{\epsilon''} A'' \to 0$. Then there is a projective resolution $\mathbf{P} : P_\bullet \xrightarrow{\epsilon} A \to 0$ of A and chain maps $f : \mathbf{P}' \to \mathbf{P}$ and $g : \mathbf{P} \to \mathbf{P}''$ such that the sequence

$$0 \to \mathbf{P}' \to \mathbf{P} \to \mathbf{P}'' \to 0$$

is exact.

Speaking in terms of diagrams, what we are saying is that, given diagram

$$\begin{array}{ccccccccc}
 & & \cdots & & & & \cdots & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & P'_0 & & & & P''_0 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{\phi} & A & \xrightarrow{\psi} & A'' & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

we can find projective resolution $P_\bullet$, f and g so we get

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'_1 & \xrightarrow{f_1} & P_1 & \xrightarrow{g_1} & P''_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'_0 & \xrightarrow{f_0} & P_0 & \xrightarrow{g_0} & P''_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon'} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon''} & & \\
0 & \longrightarrow & A' & \xrightarrow[\phi]{} & A & \xrightarrow[\psi]{} & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & \bullet & & 0 & &
\end{array}$$

with all the rows exact. Observe here if $P'_n \to P_n \to P''_n$, then it is necessary that $P_n \cong P'_n \oplus P''_n$, i.e. this short exact sequence splits.

Proof. By induction, it suffices to complete

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & K_0' & & & & K_0'' & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & P_0' & & & & P_0'' & & \\
 & & \downarrow{\scriptstyle \epsilon'} & & & & \downarrow{\scriptstyle \epsilon''} & & \\
0 & \longrightarrow & A' & \xrightarrow{\phi} & A & \xrightarrow{\psi} & A'' & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

where the rows and columns are exact and P_0', P_0'' are projective. Define $P_0 = P_0' \oplus P_0''$, $f_0 : P_0' \to P_0$ by $x' \mapsto (x', 0)$ and $g_0 : P_0 \to P_0''$ by $(x', x'') \mapsto x''$. It is clear P_0 is projective and

$$0 \to P_0' \xrightarrow{f_0} P_0 \xrightarrow{g_0} P_0'' \to 0$$

is exact. Since P_0'' is projective, there is a map $\sigma : P_0'' \to A$ with $p\sigma = \epsilon'$. Define $\epsilon : P_0 \to A$ by

$$\epsilon(x', x'') = \phi\epsilon'(x') + \sigma(x'')$$

One verify this does the job and the entire diagram commutes and all the rows are exact.

We also have a dual result about injective resolutions.

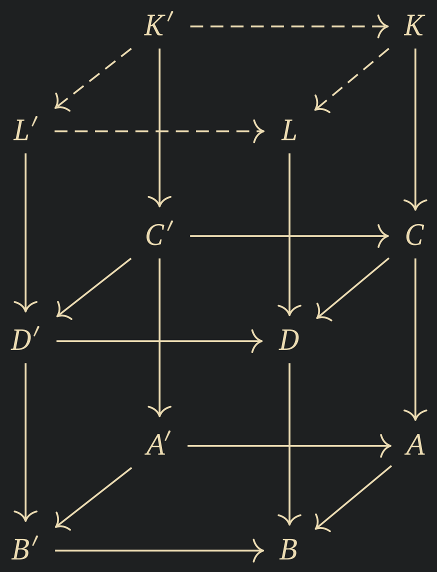
Proposition 11.15

Given short exact sequence $0 \to A' \xrightarrow{\phi} A \xrightarrow{\psi} A'' \to 0$ in abelian category $\mathcal{A}$, and injective resolutions $0 \to A' \to I'^{\bullet}$ and $0 \to A'' \to I''^{\bullet}$, then we get injective resolution $I^{\bullet}$ of A, $f : I'^{\bullet} \to I$, and $g : I^{\bullet} \to I''^{\bullet}$, such that $0 \to I'^{\bullet} \xrightarrow{f} I^{\bullet} \xrightarrow{g} I''^{\bullet} \to 0$ is exact.

Finally, we need a generalization of Horseshoe lemma for chain maps of exact sequences.

Lemma 11.16

Consider the commutative cube



in which L', K', L, K are kernels of the arrows. Then the dashed arrows exist and every new square commutes.

Proof. Existence of dashed arrows is easy. It remains to show commutativity of the top dashed square. To that end, just note $K' \to L' \to L \to D$ and $K' \to K \to L \to D$ coincide as one should verify, but $L \to D$ is monic, and thus we get $K' \to L' \to L$ equal $K' \to K \to L$.

Proposition 11.17

Assume we have commutative diagram of modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{i} & A & \xrightarrow{s} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & B' & \xrightarrow[j]{} & B & \xrightarrow[t]{} & B'' & \longrightarrow & 0 \end{array}$$

Also suppose we have projective resolutions $\mathbf{P}', \mathbf{P}'', \mathbf{Q}', \mathbf{Q}''$ of the corners A', A'', B', B'', respectively, and chain maps $F' : \mathbf{P}' \to \mathbf{Q}'$ over f', and $F'' : \mathbf{P}'' \to \mathbf{Q}''$ over f''. Then there exist projective resolutions $\mathbf{P}$ of A and $\mathbf{Q}$ of B, and a chain map $F : \mathbf{P} \to \mathbf{Q}$ over f giving commutative diagram of complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{P}' & \longrightarrow & \mathbf{P} & \longrightarrow & \mathbf{P}'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F''} & & \\ 0 & \longrightarrow & \mathbf{Q}' & \longrightarrow & \mathbf{Q} & \longrightarrow & \mathbf{Q}'' & \longrightarrow & 0 \end{array}$$

We will not prove this, but mention the proof is 3D diagram chasing. Clearly we should expect the dual version to be true as well, which we will just call it the injective version of Proposition 11.17.

12 Derived Functors

Now let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories, and let $T : \mathcal{C} \to \mathcal{D}$ be an additive functor.

Assume $\mathcal{C}$ has enough injectives. For any $A \in \mathcal{C}$, if $0 \to A \xrightarrow{\epsilon} I^\bullet$ is an injective resolution of A, then if we apply T to $I^\bullet$, then we obtain the cochain complex

$$0 \to T(I^0) \xrightarrow{T(d^0)} T(I^1) \xrightarrow{T(d^1)} T(I^2) \to \ldots \to T(I^n) \xrightarrow{T(d^n)} T(I^{n+1}) \to \ldots \qquad \text{(Eq. 12.1)}$$

denoted $T(I^\bullet)$. If $T : \mathcal{C} \to \mathcal{D}$ is a contravariant functor and if we apply T to $I^\bullet$ we obtain the chain complex

$$\ldots \to T(I^{n+1}) \xrightarrow{T(d^n)} T(I^n) \to \ldots \to T(I^1) \xrightarrow{T(d^0)} T(I^0) \to 0 \qquad \text{(Eq. 12.2)}$$

also denoted $T(I^\bullet)$.

Now assume $\mathcal{C}$ has enough projectives. For any $A \in \mathcal{C}$, if $P_\bullet \xrightarrow{\epsilon} A \to 0$ is a projective resolution of A, then if we apply T to $P_\bullet$ we get chain complex

$$\ldots \to T(P_n) \xrightarrow{T(d_n)} T(P_{n-1}) \to \ldots \to T(P_1) \xrightarrow{T(d_1)} T(P_0) \to 0 \qquad \text{(Eq. 12.3)}$$

denoted $T(P_\bullet)$. If T is contravariant functor and if we apply T to $P_\bullet$ we get cochain complex

$$0 \to T(P_0) \xrightarrow{T(d_1)} T(P_1) \to \ldots \to T(P_{n-1}) \xrightarrow{T(d_n)} T(P_n) \to \ldots \qquad \text{(Eq. 12.4)}$$

also denoted $T(P_\bullet)$.

Definition 12.1

Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories, $T : \mathcal{C} \to \mathcal{D}$ an additive functor.

1. Assume $\mathcal{C}$ has enough injectives, and we are in the case of Eq. 12.1, then we define the corresponding ***right derived functor*** is defined by

$$R^n T(I^\bullet) := H^n(T(I^\bullet))$$

2. Assume $T : \mathcal{C} \to \mathcal{D}$ is contravariant and $\mathcal{C}$ has enough injective, and we are in the case of Eq. 12.2, then the corresponding ***left derived functor*** is defined by

$$L_n T(I^\bullet) := H_n(T(I^\bullet))$$

3. Assume $\mathcal{C}$ has enough projective, and we are in the case of Eq. 12.3, then the corresponding ***left derived functor*** is defined by

$$L_n T(P_\bullet) = H_n(T(P_\bullet))$$

4. Assume $T : \mathcal{C} \to \mathcal{D}$ is contravariant and $\mathcal{C}$ has enough projective, and we are in the case of Eq. 12.4, then the corresponding ***right derived functor***

is defined by

$$R^nT(P_\bullet) = H^n(T(P_\bullet))$$

Observe the naming scheme here is solely based on whether the resulting complex is cochain complex or chain complex, i.e. if we have a chain complex after applying T, then we get a left derived functor, and otherwise right derived functor.

As the name suggests, those things are actually functors. To be exact, they are from either the category of injective resolutions or projective resolutions, to the category $\mathcal{D}$.

Let us describe how those functors act on morphisms.

Suppose we have $f : A \to B$ and injective resolutions $I^\bullet$ and $I'^\bullet$ of A and B respectively, i.e. we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle f} & & & & & & \\ 0 & \longrightarrow & B & \longrightarrow & I'^0 & \longrightarrow & I'^1 & \longrightarrow & \ldots \end{array}$$

Then by Proposition 11.12, we can find a lift $g : I^\bullet \to I'^\bullet$. Since T is a functor, $T(g)$ is a chain map from $T(I^\bullet)$ to $T(I'^\bullet)$, and in particular $T(g)$ induces a homomorphism of cohomology $H^n(T(g^n)) : H^n(T(I^\bullet)) \to H^n(T(I'^\bullet))$ for all $n \geq 0$. Furthermore, if h is another lift of f, since by Proposition 11.12 any two lifts of f are chain homotopic, say by $(s^n)_{n\geq 0}$. Since T is additive, by applying T to the equation

$$g^n - h^n = s^{n+1} \circ d_I^n + d_{I'}^{n-1} \circ s^n$$

we obtain

$$T(g^n) - T(h^n) = T(s^{n+1}) \circ T(d_I^n) + T(d_{I'}^{n-1}) \circ T(s^n)$$

which shows $(T(s^n))_{n\geq 0}$ is a chain homotopy between $T(g)$ and $T(h)$. In other word, $H^n(T(g^n)) : H^n(T(I^\bullet)) \to H^n(T(I'^\bullet))$ is independent of the lift g of f. Thus, for any arrow $f : A \to B$ in the category of injective resolutions with $0 \to A \to I^\bullet$ and $0 \to B \to I'^\bullet$, we define

$$R^nT(I^\bullet, I'^\bullet)(f) = H^n(T(g^n))$$

Similarly we define $L_nT(?,?)(f) = H_n(T(g_n))$, where the $(?,?)$ can be injective or projective resolutions, depend on T.

Theorem 12.2

Let $0 \to A \xrightarrow{\epsilon_A} I^\bullet$ and $0 \to A \xrightarrow{\epsilon'_A} I'^\bullet$ be two injective resolutions for any $A \in \mathcal{C}$. If $T : \mathcal{C} \to \mathcal{D}$ is any additive functor, then there is natural transformation

$$\eta_A^n : R^nT(I^\bullet) \to R^nT(I'^\bullet)$$

for all $n \geq 0$ that depend only on A and T. Similar results hold for other derived functors.

Proof. By Theorem 11.13 the complexes $I^\bullet$ and $I'^\bullet$ are homotopy equivalent, which means there are chain maps $g : I^\bullet \to I'^\bullet$ and $h : I'^\bullet \to I^\bullet$ both lifting Id_A such that $h \circ g$

is chain homotopic to $\mathrm{Id}_{I^\bullet}$ and $g \circ h$ is chain homotopic to $\mathrm{Id}_{I'^\bullet}$. Since T is additive, $T(h) \circ T(g)$ is chain homotopic to $\mathrm{Id}_{T(I^\bullet)}$ and $T(g) \circ T(h)$ is chain homotopic to $\mathrm{Id}_{T(I'^\bullet)}$. These chain maps induce cohomology homomorphisms for all $n \geq 0$, where

$$H^n T(h^n) \circ H^n T(g^n) = \mathrm{Id}_{T(I^\bullet)}$$

$$H^n T(g^n) \circ H^n T(h^n) = \mathrm{Id}_{T(I'^\bullet)}$$

Thus $H^n T(g^n)$ is an isomorphism of cohomology.

We still have to show this map depends only on T and A. This is because by Proposition 11.12, any two lifts g and g' of Id_A are chain homotopic, so $T(g)$ and $T(g')$ are chain homotopic. However, this implies $H^n T(g^n) = H^n T(g'^n)$. As a consequence, it is legitimate to set $\eta_A^n = H^n T(g^n)$, a well-defined isomorphism $\eta_A^n : R^n T(I^\bullet) \to R^n T(I'^\bullet)$.

It remains to show η_A^n is a natural transformation. For any $f : A \to B$ we need to show we have

$$\begin{array}{ccc} R^n T(I_A^\bullet) & \xrightarrow{\eta_A^n} & R^n T(I_A'^\bullet) \\ {\scriptstyle R^n T(I_A^\bullet, I_B^\bullet)(f)}\downarrow & & \downarrow{\scriptstyle R^n T(I_A'^\bullet, I_B'^\bullet)(f)} \\ R^n T(I_B^\bullet) & \xrightarrow{\eta_B^n} & R^n T(I_B'^\bullet) \end{array}$$

The map η_A^n is given by a lifting g_A of Id_A from $I_A^\bullet$ to $I_A'^\bullet$ and the map $R^n T(I_A^{\prime\bullet, I_B^{\prime\bullet}})(f)$ is given by a lifting h' of f from $I_A'^\bullet$ to $I_B'^\bullet$. Thus $h' \circ g_A$ is a lifting of $f \circ \mathrm{Id}_A = f$. Similarly $g_B \circ h$ is a lifting of $\mathrm{Id}_B \circ f = f$. Thus we see $T(h') \circ T(g_A)$ and $T(g_B) \circ T(h)$ are both lifts of $T(f)$, and so by Proposition 11.12 they are chain homotopic, which concludes the commutativity.

In conclude, we see R^n and L_n are indeed functors, where we map an arrow $f : \to B$ to, for example, $R^n T(I_A^\bullet, I_B^\bullet)(f)$.

Here are some basic properties of derived functors.

Proposition 12.3

Let $\mathcal{C}, \mathcal{D}$ be abelian categories and $T : \mathcal{C} \to \mathcal{D}$ be additive functor.

1. *If T is left exact then $R^0 T$ is naturally isomorphic to T. If T is right exact and contravariant then $L_0 T$ is naturally isomorphic to T.*
2. *If T is right exact then $L_0 T$ is naturally isomorphic to T. If T is left exact and contravariant then $R^0 T$ is naturally isomorphic to T.*

Proof. We only do (1). Suppose $0 \to A \xrightarrow{\epsilon} I^\bullet$ is an injective resolution of A. Since T is left exact we have

$$0 \to T(A) \to T(I_0) \to T(I_1)$$

Since $T(\epsilon)$ is injective, $T(A) \cong \operatorname{im} T(\epsilon) = \ker T(d^0)$. Thus the chain complex $T(I^\bullet)$ gives

$$R^0 T(A) = H^0(T(I^\bullet)) = \ker T(d^0)$$

so $T(A)$ is isomorphic to $R^0T(A)$.

Example 12.4

Take the functor T to be $T_B(A) := \mathrm{Hom}(A,B)$. Then we see

$$\mathrm{Ext}^n_R(A,B) = (R^nT_B)(A)$$

with $\mathrm{Ext}^0_R(A,B) \cong \mathrm{Hom}(A,B)$ as $\mathrm{Hom}(-,B)$ is left exact. Similarly, if we take $T_B(A) = A \otimes_R B$, then we get back to Tor, and since this functor T is right exact, $\mathrm{Tor}^R_0(A,B) = A \otimes B$.

Proposition 12.5

Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories, and $T : \mathcal{C} \to \mathcal{D}$ be additive functor. Then:

1. *For every injective object I, $R^nT(I) = (0)$ for all $n \geq 1$, and $T(I)$ is isomorphic to $R^0T(I)$. If T is contravariant, $L_nT(I) = (0)$ for all $n \geq 1$ and $L_0T(I) \cong T(I)$.*
2. *For every projective object P, $L_nT(P) = 0$ for $n \geq 1$ and $T(P) \cong L_0T(I)$. If T is contravariant, $R^nT(P) = 0$ for $n \geq 1$ and $T(P) \cong R^0T(P)$.*

Proof. We only show (1). If I is injective, then consider

$$0 \to I \xrightarrow{\mathrm{Id}} I \to 0$$

This gives $0 \to T(I) \to 0$ which shows $R^0T(I) = H^0(T(I^\bullet)) = T(I)$ and 0 for $n \geq 1$.

The following result shows that short exact sequence gives long exact sequence of cohomology or homology. A similar result also holds for all other cases.

Theorem 12.6

Let $\mathcal{C}$ be abelian with enough injectives. Let $0 \to A' \to A \to A'' \to 0$ be an eaxct sequence in $\mathcal{C}$, and $T : \mathcal{C} \to \mathcal{D}$ an additive left-exact functor.

1. *For every $n \geq 0$, there is a map $(R^nT)(A'') \xrightarrow{\delta^n} (R^{n+1}T)(A')$ and makes the cochain complex*

$$0 \to T(A') \to T(A) \to \ldots \to (R^nT)(A'') \xrightarrow{\delta^n} (R^{n+1}T)(A') \to \ldots$$

exact.

2. *If we have*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

with all rows eaxct, then the induced diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & T(A') & \longrightarrow & \dots & \longrightarrow & R^nT(A'') & \xrightarrow{\delta^n_A} & R^{n+1}T(A') & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T(B') & \longrightarrow & \dots & \longrightarrow & R^nT(B'') & \xrightarrow[\delta^n_B]{} & R^{n+1}T(A') & \longrightarrow & \dots \end{array}$$

commutes.

We will prove this after some examples, where from places to places we used this result implicitly.

In what follows we will spend a little bit time compute some functors, before proceed to more theory.

Example 12.7

Let R be commutative, then $\text{Tor}_1^R(R/xR, M) = \{m \in M : xm = 0\}$, where x is not zero-divisor. To see this, consider the resolution

$$0 \to Rx \to R \to R/Rx \to 0$$

and so we get

$$0 \to \text{Tor}_1^R(M, R/Rx) \to M \otimes Rx \to M \to M \otimes R/Rx \to 0$$

where $\ker(M \otimes Rx \to M) \cong \text{Tor}_1^R(M, R/Rx) = \{m \in M : xm = 0\}$, where we used the identification R-module we see $Rx \cong R$ and so $M \otimes Rx \cong M \otimes R \cong M$.

Example 12.8

We have $\text{Tor}_1^R(R/I, R/J) = I \cap J/IJ$, which in some sense measures how far away we are from "nice" intersection. Consider

$$0 \to J \to R \to R/J \to 0$$

This gives

$$0 \to \text{Tor}_1(R/I, R/J) \to R/I \otimes J \to R/I \to R/(I+J) \to 0$$

where we recall $R/I \otimes J \cong J/IJ$. Thus we get

$$0 \to \text{Tor}_1 \to J/IJ \to R/I \to R/(I+J) \to 0$$

where the $J \to IJ \to R/I$ is given by $j + IJ \mapsto j + I$. In other word, the kernel is given by $I \cap J/IJ$, and it is equal Tor_1.

As a result of this, we see if $I+J=R$ then $IJ=I\cap J$. Indeed, just note we have

$$0\to \mathrm{Tor}_1\to J/IJ\to R/I\to R/(I+J)\to 0$$

where $R/(I+J)=0$.

Example 12.9

Let R be local Noetherian, with $k=R/\mathfrak{m}$ the residue field. Then we have a minimal resolution of M (meaning $\mathrm{im}(\phi_i)\subseteq \mathfrak{m}R^{b_i-1}$), where the resolution is

$$\ldots\to R^{b_2}\xrightarrow{\phi_2} R^{b_1}\xrightarrow{\phi_1} R^{b_0}\to M\to 0$$

Then $\mathrm{Tor}_i(k,M)\cong k^{b_i}$, and the b_i are called the Betti numbers of M.

Example 12.10

Let us compute Tor and Ext for f.g. $\mathbb{Z}$-modules. To that end, note

$$G\cong \bigoplus \mathbb{Z}\oplus\bigoplus_i \mathbb{Z}/n_i\mathbb{Z}$$

and thus by functoriality, it suffices to compute Ext and Tor for the pairs $(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})$, $(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z})$ $(\mathbb{Z},\mathbb{Z})$.

First, for $\mathrm{Ext}(\mathbb{Z},\mathbb{Z})$, since $\mathbb{Z}$ itself is projective, we are done by one of the above result, i.e. $\mathrm{Ext}^n(\mathbb{Z},\mathbb{Z})=0$ for $n\geq 1$.

Next, to compute $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})$, consider

$$0\to\mathbb{Z}\xrightarrow{\cdot n}\mathbb{Z}\to\mathbb{Z}/n\mathbb{Z}\to 0$$

and apply Hom we get

$$0\leftarrow \mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})\leftarrow\mathbb{Z}\xleftarrow{\cdot n}\mathbb{Z}\leftarrow \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})\leftarrow 0$$

This shows $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})=\mathrm{coker}(\mathbb{Z}\xrightarrow{\cdot n}\mathbb{Z})=\mathbb{Z}/n\mathbb{Z}$.

For $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z})$, consider

$$0\to\mathbb{Z}\xrightarrow{\cdot n}\mathbb{Z}\to\mathbb{Z}/n\mathbb{Z}\to 0$$

Applying Hom we get

$$0\to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z})\to\mathbb{Z}/m\mathbb{Z}\xrightarrow{\cdot n}\mathbb{Z}/m\mathbb{Z}\to\mathrm{Ext}^1\to 0$$

Taking homology groups this gives

$$\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z})\cong(\mathbb{Z}/m\mathbb{Z})/n(\mathbb{Z}/m\mathbb{Z})\cong\mathbb{Z}/d\mathbb{Z}$$

with $d=\gcd(n,m)$.

We will end with computing $\operatorname{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$, and leave the $\operatorname{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ as exercise. To that end, consider the same projective resolution

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

This gives

$$0 \to \mathbb{Z} \otimes \mathbb{Z}/m \to \mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \to 0$$

which is the same as

$$0 \to \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/(n+m)\mathbb{Z} \to 0$$

Now take homology we get $\operatorname{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$, where d is the gcd of m, n.

Proof of Theorem 12.6. We have injective resolutions $0 \to A' \xrightarrow{\epsilon'} I^\bullet_{A'}$ and $0 \to A'' \xrightarrow{\epsilon''} I^\bullet_{A''}$. Thus by Horseshoe Lemma 11.15 we get

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^1_{A'} & \longrightarrow & I^1_A & \longrightarrow & I^1_{A''} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^0_{A'} & \longrightarrow & I^0_A & \longrightarrow & I^0_{A''} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This gives short exact sequence

$$0 \to I^\bullet_{A'} \to I^\bullet_A \to I^\bullet_{A''} \to 0$$

Now, to compute $R^n T$, we actually already have an injective resolution $\hat{I}^\bullet_A$ of A, but by Theorem 12.2 we see those two will yield the same derived functor, and so we might as well assume our injective resolution is induced by the Horseshoe Lemma 11.15. Now apply T, we get

$$0 \to T(I^\bullet_{A'}) \to T(I^\bullet_A) \to T(I^\bullet_{A''}) \to 0$$

This is because $0 \to I^n_{A'} \to I^0_A \to I^0_{A''} \to 0$ splits and T is additive. However, now by (cohomology version of) Zig-Zag Lemma 10.8, we get the desired result. To prove naturality, just use the injective version of Proposition 11.17.

A similar result holds for left derived functors, in the appropriate setting.

To conclude this section, we will record some properties of Tor and Ext.

Proposition 12.11

Let R be a ring, M an R-module. Then the following are equivalent:

1. *M is flat over R*
2. $\mathrm{Tor}_i^R(M,-)$ *is zero for all* $i \geq 1$
3. $\mathrm{Tor}_1^R(M,-)$ *is zero*
4. *For all ideals* $I \subseteq R$, $\mathrm{Tor}_1^R(M,R/I) = 0$
5. *For all f.g. ideals* $I \subseteq R$, $\mathrm{Tor}_1^R(M,R/I) = 0$

Proposition 12.12

Let R be Noetherian, M,N be finite R-modules. Then $\mathrm{Tor}_p^R(M,N)$ is finite R-module for all p.

Proposition 12.13

Let P be R-module. Then the following are equivalent:

1. *P is projective over R*
2. $\mathrm{Ext}^1(P,-) = 0$
3. $\mathrm{Ext}^i(P,-) = 0$ *for all* $i \geq 1$

13 δ-Functors

We will conclude this section by T-acyclic resolutions (this is used to compute derived functors) and universal δ-functors (this is a little bit hard to explain, but it is a type of uniqueness result).

Definition 13.1

Let $T : \mathcal{C} \to \mathcal{D}$ be left exact additive functor. An object $J \in \mathcal{C}$ a ***(right) T-acyclic*** if $R^n T(J) = 0$ for all $n \geq 1$.

Proposition 13.2

If $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ *is exact and T is left exact, then* $\ker T(g) \cong T(\ker g)$.

Proof. Exercise.

Proposition 13.3

Given left eaxct additive functor $T : \mathcal{C} \to \mathcal{D}$, for any $A \in \mathcal{C}$, suppose there is exact sequence

$$0 \to A \xrightarrow{\epsilon} J^0 \xrightarrow{d^0} J^1 \xrightarrow{d^1} J^2 \to \dots \tag{Eq. 13.1}$$

in which every J^n is right T-acyclic (we denote the $0 \to J^0 \to J^1 \to \dots$ as $J_A^\bullet$ and call it a right T-acyclic resolution). Then for all $n \geq 0$, we have isomorphism between $R^n T(A)$ and $H^n(T(J_A^\bullet))$.

Proof. Since Eq. 13.1 is exact and T is left-exact, we get exact sequence

$$0 \to T(A) \to T(J^0) \to T(J^1)$$

which implies

$$R^0 T(A) \cong T(A) \cong \ker T(d^0) = H^0(T(J_A^\bullet))$$

Let $K^n = \ker d^n$ for all $n \geq 1$, then Eq. 13.1 implies $\operatorname{im} d^n = \ker d^{n+1} = K^{n+1}$ and the surjection $p^n : J^n \to K^{n+1}$ has kernel K^n so we have short exact sequence

$$0 \to K^n \to J^n \xrightarrow{p^n} K^{n+1} \to 0 \tag{Eq. 13.2}$$

for all $n \geq 1$. We also have

$$0 \to A \to J^0 \xrightarrow{p^0} K^1 \to 0 \tag{Eq. 13.3}$$

If we denote the injection of K^{n+1} into J^{n+1} by ϵ^{n+1}, then we get

$$d^n = \epsilon^{n+1} \circ p^n$$

Now apply T we get

$$T(d^n) = T(\epsilon^{n+1}) \circ T(p^n)$$

Since ϵ^{n+1} is injective, $0 \to K^{n+1} \xrightarrow{\epsilon^{n+1}} J^{n+1} \xrightarrow{d^{n+1}} J^{n+2}$ is exact, and since T is left exact, we see $0 \to T(K^{n+1}) \to T(J^{n+1}) \to T(J^{n+2})$ is also exact. This means the restriction of $T(\epsilon^{n+1})$ to $\operatorname{im} T(p^n)$ is an isomorphism onto the image of $T(d^n)$. Thus we see

$$\operatorname{im} T(d^n) \cong \operatorname{im} T(p^n), \quad n \geq 0$$

By definition of $K^n = \ker d^n$, we have

$$0 \to K^n \to J^n \xrightarrow{d^n} J^{n+1}$$

so by Proposition 13.2 we get

$$\ker T(d^n) \cong T \ker(d^n)$$

Now apply Theorem 12.6 to Eq. 13.3 we get long exact sequence begins with

$$0 \to T(A) \to T(J^0) \to T(K^1) \to R^1 T(A) \to R^1 T(J^0) = 0$$

which gives

$$\begin{aligned} R^1 T(A) &\cong T(K^1)/\operatorname{im} T(p^0) = T(\ker d^1)/\operatorname{im} T(p^0) \\ &\cong \ker T(d^1)/\operatorname{im} T(d^0) \\ &= H^1(T(J_A^\bullet)) \end{aligned}$$

This concludes $R^1T(A) \cong H^1(T(J_A^\bullet))$.

It remains to prove $n \geq 2$ case. For that, note by Theorem 12.6 on Eq. 13.3, we also get eaxct sequence

$$R^{n-1}T(J^0) \to R^{n-1}T(K^1) \to R^nT(A) \to R^nT(J^0)$$

where $R^{n-1}T(J^0) = R^nT(J^0) = 0$ since $J_A^\bullet$ is right T-acyclic resolution. Therefore we get

$$R^{n-1}T(K^1) \cong R^nT(A), \quad n \geq 2$$

Now use the long exact sequence (from Theorem 12.6) induced by Eq. 13.2, we get

$$R^{n-i-1}T(J^i) \to R^{n-i-1}T(K^{i+1}) \to R^{n-i}T(K^i) \to R^{n-i}T(J^i)$$

and since J^i is T-acyclic, we get

$$R^{n-i-1}T(K^{i+1}) \cong R^{n-i}T(K^i), \quad 1 \leq i \leq n-2$$

Now apply induction we conclude

$$R^{n-1}T(K^1) \cong R^1T(K^{n-1}), \quad n \geq 2$$

where we know $R^{n-1}T(K^1) \cong R^nT(A)$, i.e. we have

$$R^nT(A) \cong R^{n-1}T(K^1) \cong R^1T(K^{n-1})$$

The long exact sequence applied to Eq. 13.2 gives

$$T(J^{n-1}) \to T(K^n) \to R^1T(K^{n-1}) \to R^1T(J^{n-1}) = 0$$

and thus by first isomorphism theorem and Proposition 13.2 we get

$$\begin{aligned} R^nT(A) &\cong R^1T(K^{n-1}) \\ &\cong T(K^n)/\operatorname{im} T(p^{n-1}) \\ &\cong T(\ker d^n)/\operatorname{im} T(p^{n-1}) \\ &\cong \ker T(d^n)/\operatorname{im} T(d^{n-1}) \\ &= H^n(T(J_A^\bullet)) \end{aligned}$$

This concludes the proof.

The above result can also be proved using the following proposition.

Proposition 13.4

Let $T : \mathcal{C} \to \mathcal{D}$ be additive left eaxct functor. For any T-acyclic exact cochain $X^\bullet$: $0 \to X^0 \to X^1 \to \ldots$, we have $T(X^\bullet)$ is also exact.

All the above results also holds for left T-acyclic resolutions and left derived functors L_nT.

Definition 13.5

Given two abelian categories $\mathcal{C}$ and $\mathcal{D}$, a ***δ-functor*** consists of a countable family $T = (T^n)_{n\geq 0}$ of additive functors $T^n : \mathcal{C} \to \mathcal{D}$, and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{C}$ and $n \geq 0$, we have a map

$$T^n(A'') \xrightarrow{\delta^n} T^{n+1}(A')$$

such that:

1. The sequence $0 \to T^0(A') \to \ldots \to T^n(A'') \xrightarrow{\delta^n} T^{n+1}(A') \to \ldots$ is eaxct.
2. If we have commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

with all the rows eaxct then the induced diagram

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & T^n(A) & \longrightarrow & T^n(A'') & \longrightarrow & T^{n+1}(A') & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & T^n(B) & \longrightarrow & T^n(B'') & \longrightarrow & T^{n+1}(B') & \longrightarrow & \ldots \end{array}$$

also commutes.

Clearly the left and right derived functors are δ-functors. What is non-trivial is that every δ-functor is isomorphic to some derived functors (such δ-functor is called universal).

The collection of δ functors from $\mathcal{C}$ to $\mathcal{D}$ forms a category, and the morphism is the obvious one.

Definition 13.6

A δ-functor $T = (T^n)_{n\geq 0}$ is ***universal*** if for every δ-functor $S = (S^n)$ and every natural transformation $\phi : T^0 \to S^0$, there is a unique $\eta : T \to S$ such that $\eta^0 = \phi$.

As you would expect, in the above situation, we say η ***lift*** ϕ, if for every short exact sequence $0 \to A' \to A \to A'' \to 0$, we have

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & T^0(A'') & \longrightarrow & T^1(A') & \longrightarrow & T^1(A) & \longrightarrow & T^1(A'') & \xrightarrow{\delta_T^1} & T^2(A') & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \phi_{A''}} & & \downarrow{\scriptstyle \eta^1_{A'}} & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & S^0(A'') & \longrightarrow & S^1(A') & \longrightarrow & S^1(A) & \longrightarrow & S^1(A'') & \xrightarrow{\delta_S^1} & S^2(A') & \longrightarrow & \ldots \end{array}$$

Proposition 13.7

If $S = (S^n)$ and $T = (T^n)$ are both universal δ-functors and there is an isomorphism $\phi : S^0 \to T^0$. Then there is a unique isomorphism $\eta : S \to T$ lifting ϕ.

In short, this result tells us a universal δ-functor is completely determined by the T^0 component. Next, we show that δ-functors exists.

Theorem 13.8

Assume $\mathcal{C}$ has enough injectives. For every additive left exact functor $T : \mathcal{C} \to \mathcal{D}$, the family $(R^n T)_{n\geq 0}$ of right derived functors of T is a δ-functor. Furthermore, T is isomorphic to $R^0 T$.

Proof. This is just restate things we proved above.

Dual to δ-functors, we also have a notion of ∂-functors, which mimics $L_n T$. We leave as an exercise for the readers to figure out the definitions of ∂-functors and the morphisms between them, as well as the dual results to the above.

The next goal is to show the derived functors are universal δ-functors. To that end, we will need the notion of erasable/effaçable functors. The term effaçable functors are used by Grothendieck, but we will stick to erasable.

Definition 13.9

An additive functor $T : \mathcal{C} \to \mathcal{D}$ is ***erasable*** if for every object $A \in \mathcal{C}$ there is $M_A \in \mathcal{C}$ with monic $u : A \to M_A$ such that $T(u) = 0$.

If T is erasable, then we always have $T(M_A) = 0$ in $\mathcal{D}$. Also, we have a dual notion of ***co-erasable***, where we require every $A \in \mathcal{C}$ to have a $M_A \in \mathcal{C}$ and epic $u : M_A \to A$ such that $T(u) = 0$.

In many cases, the M_A can be in fact choosen to be injective (or in co-erasable case choosen to be projective). In this case we say T is erasable by injective (or co-erasable by projective). However, this is not always desirable.

Proposition 13.10

1. *Suppose $\mathcal{C}$ is abelian with enough injectives. For every additive left exact $T : \mathcal{C} \to \mathcal{D}$, the right derived functor $R^n T$ is erasable by injectives for all $n \geq 1$.*
2. *Suppose $\mathcal{C}$ is abelian with enough projective. For every additive right exact $T : \mathcal{C} \to \mathcal{D}$, the left derived functors $L_n T$ are co-erasable by projectives for all $n \geq 1$.*

Definition 13.11

Let $\mathcal{C}$ be abelian. For every $A \in \mathcal{C}$, an ***injective erasing*** of A is a monic $u : A \to M$ such that for every monic $g : B \to C$ and any map $f : B \to A$, there is some map $\tilde{f} : C \to M$ making the following diagram commute

$$\begin{array}{ccccc} 0 & \longrightarrow & B & \xrightarrow{g} & C \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \tilde{f}} \\ 0 & \longrightarrow & A & \xrightarrow{u} & M \end{array}$$

If $\mathcal{C}$ has enough injective then $u : A \to I$ with I injective is always an injective erasing of A.

The following result shows some relationship between erasibility and injective erasing.

Proposition 13.12

Suppose $T : \mathcal{C} \to \mathcal{D}$ is additive functor.

1. *If T is erasable then for any injective erasing $u : A \to M$, $T(u) = 0$*
2. *If every $A \in \mathcal{C}$ has injective erasing then T is erasable iff $T(u) = 0$ for all injective erasing $u : A \to M$*
3. *If T is erasable then $T(I) = 0$ for all injective object I*
4. *If $\mathcal{C}$ has enough injectives, then T is erasable iff $T(I) = 0$ for all injective object I*

Proof. Mostly check definitions.

Now we are ready to state one of the big result on δ-functors.

Theorem 13.13: Grothendieck

Let $T = (T^n)$ be a δ-functor between $\mathcal{C}$ and $\mathcal{D}$. If every $A \in \mathcal{C}$ has injective erasing $v : A \to M_A$ such that $T^n(v) = 0$ for all $n \geq 1$, then T is a universal δ-functor.

Proof. The proof presented here is not due to Grothendieck. It can be found in Jean Gallier's book, where it states this proof is essentially due to Steve Shatz.

We will do induction on n, and only do the case $n = 1$, as the rest is similar.

Step 1: Construction of the lift map u_1.

Let $S = (S^n)$ be another δ-functor and let $u_0 : T^0 \to S^0$ be a given map of functors. If A is an object of $\mathcal{C}$, injective erasing of A for T^1 shows we have exact sequence

$$0 \to A \xrightarrow{v} M_A \xrightarrow{p} A'' \to 0 \qquad \text{(Eq. 13.4)}$$

with $A'' = \operatorname{coker}(v)$, such that $\delta^0_{T^0}$ in the induced sequence

$$T^0(M_A) \to T^0(A'') \to T^1(A) \to T^1(M_A)$$

is surjective (since $T^1(v) = 0$). Since T is a δ-functor, we have commutative diagram

$$\begin{array}{ccccccc}
T^0(M_A) & \longrightarrow & T^0(A'') & \longrightarrow & T^1(A) & \longrightarrow & T^1(M_A) \\
\downarrow{\scriptstyle u_0(M_A)} & & \downarrow{\scriptstyle u_0(A'')} & & \downarrow{\scriptstyle u_1} & & \\
S^0(M_A) & \longrightarrow & S^0(A'') & \longrightarrow & S^1(A) & &
\end{array}$$

Since $\ker \delta^0_{T^0} = \operatorname{im} T^0(p)$, since the left square commutes

$$u_0(A'') \circ T^0(p) = S^0(p) \circ u_0(M_A)$$

and since the bottom row is exact, we get

$$\delta^0_{S^0} \circ u_0(A'') \circ T^0(p) = \delta^0_{S^0} \circ S^0(p) \circ u_0(M_A) = 0$$

This shows

$$\ker \delta^0_{T^0} \subseteq \ker(\delta^0_{S^0} \circ u_0(A''))$$

Since $\delta^0_{T^0}$ is surjective, we define $u_1 = T^1(A) \to S^1(A)$ as follows: for any $a \in T^1(A)$, pick any $b \in T^0(A'')$ such that $a = \delta^0_{T^0}(b)$, and set

$$u_1(a) = (\delta^0_{S^0} \circ u_0(A''))(b) \qquad \text{(Eq. 13.5)}$$

This map is well-defined, because if $a = \delta^0_{T^0}(b')$ for some $b' \in T^0(A'')$, then $\delta^0_{T^0}(b) = \delta^0_{T^0}(b')$, so $\delta^0_{T^0}(b - b') = 0$, i.e. $b - b' = c \in \ker \delta^0_{T^0} \subseteq \ker(\delta^0_{S^0} \circ u_0(A''))$. This shows $b' = b + c$ with $c \in \ker(\delta^0_{S^0} \circ u_0(A''))$, which implies

$$\begin{aligned}
(\delta^0_{S^0} \circ u_0(A''))(b') &= (\delta^0_{S^0} \circ u_0(A''))(b + c) \\
&= (\delta^0_{S^0} \circ u_0(A''))(b)
\end{aligned}$$

Thus, the map $u_1 : T^1(A) \to S^1(A)$ making the second square commute is uniquely defined. It remains to check u_1 has the required properties and it does not depend on the choice of the exact sequence Eq. 13.4

Step 2: Independence of the choice $A \to M_A$.

Suppose we have $0 \to \tilde{A} \xrightarrow{\tilde{v}} \widetilde{M_A} \xrightarrow{\tilde{p}} \widetilde{A''} \to 0$ where $\tilde{v}$ is injective erasing, and $\widetilde{A''} = \operatorname{coker}(\tilde{v})$. By hypothesis, $T(\tilde{v}) = 0$. Assume we have $g : A \to \tilde{A}$. Since $\tilde{v}$ is injective erasing and v is monic, there is a map θ extending $\tilde{v} \circ g$ making the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & M_A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \theta} & & & & \\
0 & \longrightarrow & \tilde{A} & \longrightarrow & \widetilde{M_A} & \longrightarrow & \widetilde{A''} & \longrightarrow & 0
\end{array}$$

commute. Now the diagram

$$\begin{array}{ccccccc}
A & \longrightarrow & M_A & \longrightarrow & A'' & \longrightarrow & 0 \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \theta} & & & & \\
\tilde{A} & \longrightarrow & \widetilde{M_A} & \longrightarrow & \widetilde{A''} & \longrightarrow & 0
\end{array}$$

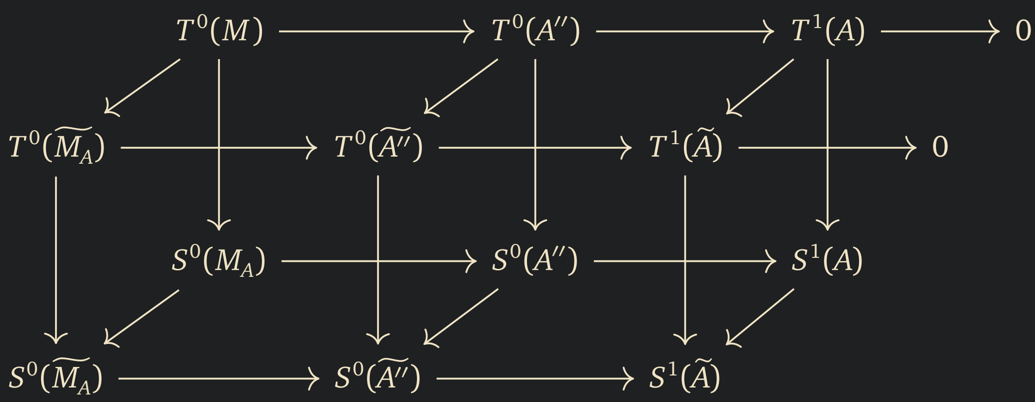
is similar to the commutative diagram used in the construction of u_1 in Step 1, and it has exact row, so the same argument shows there is a $\overline{\theta} : A'' \to \widetilde{A''}$ making the above diagram commute. Now Theorem 12.6 applied to the above with T and S yields the two commutative diagram

$$\begin{array}{ccccccc}
T^0(M_A) & \longrightarrow & T^0(A'') & \longrightarrow & T^1(A) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle T^1(g)} & & \\
T^0(\widetilde{M_A}) & \longrightarrow & T^0(\widetilde{A''}) & \longrightarrow & T^1(\tilde{A}) & \longrightarrow & 0
\end{array}$$

since $T^1(v) = 0$ and $T^1(\tilde{v}) = 0$, and

$$\begin{array}{ccccc}
S^0(M_A) & \longrightarrow & S^0(A'') & \longrightarrow & S^1(A) \\
\downarrow & & \downarrow & & \downarrow \\
S^0(\widetilde{M_A}) & \longrightarrow & S^0(\widetilde{A''}) & \longrightarrow & S^1(\tilde{A})
\end{array}$$

We can also apply Theorem 12.6 to the two commutative diagrams involved in the construction of u_1 and $\tilde{u_1}$ as in Step 1. This gives a diagram



where all the top, bottom, front and back squares are commutative, and the two left hand vertical squares also commute by naturality of u_0. Since $\delta^0_{T^0(A'')} : T^0(A'') \to T^1(A)$ is surjective, if we can show the two compositions $T^0(A'') \to T^1(A) \to T^1(\tilde{A}) \to S^1(\tilde{A})$ and $T^0(A'') \to T^1(A) \to S^1(A) \to S^1(\tilde{A})$ are equal, i.e. we have

$$\tilde{u_1} \circ T^1(g) \circ \delta^0_{T^0(A'')} = S^1(g) \circ u_1 \circ \delta^0_{T^0(A'')}$$

then we can conclude $\tilde{u_1} \circ T^1(g) = S^1(g) \circ u_1$, which concludes the entire diagram is commutative. The verification is left as an exercise, but you should use commutative of the other five faces of the rightmost cube, in the order: top, front, left, bottom, back.

Thus, at this point, we see the above 3D diagram is commutative. Now set $A = \tilde{A}$ and $g = \mathrm{Id}$, we see $\tilde{u_1} = u_1$, and thus u_1 is independent of M_A.

Step 3: Prove u_1 is functorial.

Let $g : A \to \tilde{A}$, u_1 and $\tilde{u_1}$ be as in Step 1. We need to show

$$\begin{array}{ccc}
T^1(A) & \xrightarrow{T^1(g)} & T^1(\tilde{A}) \\
\downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle \tilde{u_1}} \\
S^1(A) & \xrightarrow{S^1(g)} & S^1(\tilde{A})
\end{array}$$

is commutative. However, this is precisely the commutativity of the rightmost face in the above 3D diagram, and thus we are done.

Step 4: We need to show for any short exact sequence $0 \to A' \xrightarrow{\phi} A \xrightarrow{\psi} A'' \to 0$, the diagram

$$\begin{CD} T^0(A'') @>{\delta^0_{T^0}}>> T^1(A') \\ @V{u_0(A'')}VV @VV{u_1}V \\ S^0(A'') @>{\delta^0_{S^0}}>> S^1(A') \end{CD}$$

is commutative. Here we need to be careful as ψ is not necessarily erased, so the previous construction does not work. However, there is an injective erasing

$$0 \to A' \xrightarrow{v} M_{A'} \xrightarrow{p} X \to 0$$

and as before we get

$$\begin{CD} 0 @>>> A' @>{\phi}>> A @>{\psi}>> A'' @>>> 0 \\ @. @V{\mathrm{Id}_{A'}}VV @VV{\theta}V @VV{\tilde{\theta}}V \\ 0 @>>> A' @>>{v}> M_{A'} @>>{p}> X @>>> 0 \end{CD}$$

Since T is a δ-functor, we obtain the commutative diagram

$$\begin{CD} T^0(A'') @>>> T^1(A') \\ @V{T^0(\overline{\theta})}VV @VV{\mathrm{Id}}V \\ T^0(X) @>>> T^1(A') \end{CD}$$

We also get similar diagram for S as S is also a δ-functor. Next, since u_0 is a natural transformation, we get

$$\begin{CD} T^0(A'') @>{T^0(\overline{\theta})}>> T^0(X) \\ @V{u_0(A'')}VV @VV{u_0(X)}V \\ S^0(A'') @>{S^0(\overline{\theta})}>> S^0(X) \end{CD}$$

The construction of u_1 in Step 1 gives

$$\begin{CD} T^0(X) @>{\delta^0_{T^0}}>> T^0(A') \\ @V{u_0(X)}VV @VV{u_1}V \\ S^0(X) @>{\delta^0_{S^0}}>> S^0(A') \end{CD}$$

We leave it as an exercise to check the cube formed by the four above diagrams is a commutative cube. However, the cube is commutative implies the diagram we need to show in Step 4 is commutative, and thus we are done.

There are other conditions that tells T is universal δ-functor, which we record below.

Theorem 13.14

Let T be a δ-functor between $\mathcal{C}$ and $\mathcal{D}$.

1. *If T^n are erasable for all $n \geq 1$ then T is universal δ-functor*
2. *If $\mathcal{C}$ has enough injectives and $T^n(I) = 0$ for all injective I and all $n \geq 1$, then T is universal δ-functor.*

Theorem 13.15

Suppose $\mathcal{C}$ has enough injectives. For every additive left-exact $T : \mathcal{C} \to \mathcal{D}$, the right derived functors $(R^n T)_{n\geq 0}$ is a universal δ-functor such that $T \cong R^0 T$. Conversely, every universal δ-functor $T = (T^n)_{n\geq 0}$ is isomorphic to the right derived δ-functor $(R^n T^0)_{n\geq 0}$.

Proof. The first statement is 13.13, Proposition 13.10, and Theorem 13.8 together. The converse is Proposition 13.7.

The dual results on ∂-functors and co-erasable by projective T_n also holds, we leave it to the readers to figure out the claims.

Theorem 13.16

If $\mathcal{C}$ has enough projectives. Then for every additive right exact $T : \mathcal{C} \to \mathcal{D}$, $(L_n T)_{n\geq 0}$ is universal ∂-functor such that $T \cong L_0 T$. Conversely, every universal ∂-functor $T = (T_n)$ is isomorphic to $(L_n T_0)_{n\geq 0}$.

To conclude this section, we mention one (very important) application of universal δ-functor. The main point of δ-functors is to show two cohomology theories agree. For example, for paracompact spaces the Čech cohomology $\check{H}^*(X, \mathscr{F})$ is isomorphic to $H^*(X, \mathscr{F})$, where the second cohomology is sheaf cohomology. The way we prove this is to show both cohomologies are universal δ-functors, but they agree on the 0th pieces, i.e. $\check{H}^0(X, \mathscr{F}) = H^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$ and hence by the above results, they must agree overall.

14 Spectral Sequences

The exposition we take here mostly follow from the book "differential forms in algebraic topology", as well as course notes. Since we can embed things into R-modules, we will work with those instead of general abelian categories.

One of the motivation of spectral sequence is as follows: we want to compute cohomology $H(M)$, where M has a filtration, but cohomology is hard to compute. On the other hand, maybe the cohomology for its filtration is easy to compute (its contains smaller pieces anyway). Thus, we ask when is the cohomology of its filtration is equal the cohomology of M. This gives spectral sequence. Thus, in some sense, spectral sequence is the thing we use to approximate cohomology groups based on smaller pieces. Each step of the approximation is called a page, and the infinite page will hopefully be the original cohomology, but it is not always the case.

This construction of a spectral sequence by exact couples is due to Massey (see his paper "exact couples in algebraic topology I,II" in Ann of Math).

Definition 14.1

An ***exact couple*** (A, B) is an exact sequence of objects in abelian $\mathcal{A}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{i} & A \\ {}_{k}\nwarrow & & \swarrow_{j} \\ & B & \end{array}$$

Given exact couple (A, B) we can define $d : B \to B$ by $d = j \circ k$, and it is obvious $d^2 = jkjk = 0$ as $\operatorname{im}(j) = \ker(k)$ and so $kj = 0$. Thus, we can speak of the cohomology group $H(B) = \ker(d)/\operatorname{im}(d)$.

Construction 14.2

Let (A, B) be an exact couple, we can construct its ***derived couple***

$$\begin{array}{ccc} A' & \xrightarrow{i'} & A' \\ {}_{k'}\nwarrow & & \swarrow_{j'} \\ & B' & \end{array}$$

by:

1. $A' = i(A)$, $B' = H(B)$
2. i is induced by i, i.e. $i'(i(a)) = i^2(a)$
3. $j'(a') = ja + \operatorname{im}(d) \in \ker(d)/\operatorname{im}(d)$ if $a' = ia \in A'$
4. k' is induced by k, i.e. if $[b] = b + \operatorname{im}(d)$ lies in $H(B) = B'$, then $k'[b] = kb$. This lies in $i(A)$ because $[b] \in H(B)$ then $jkb = 0$ and hence $kb = ia$ for some $a \in A$.

Right, in the above, we need to check j' is well-defined. To that end, suppose we have $a' = ia_0$ and $a' = ia_1$ for $a_1, a_0 \in A$. Then $0 = i(a_0 - a_1)$ we see $a_0 - a_1 = kb$ for some $b \in B$. Thus $ja_0 - ja_1 = jkb = db$, which shows $j'(a')$ is well-defined up to an element in $\operatorname{im}(d)$, i.e. its well-defined in $\ker(d)/\operatorname{im}(d)$. As you would expect, this derived couple is also an exact couple, which we left as an exercise.

In the following, we are going to talk about some spectral sequences, and for that we will first define what they are.

Definition 14.3

A ***(homological) spectral sequence*** $E = \{E^r, d_r\}$ consists of a sequence of $\mathbb{Z}$-bigraded R-modules $E^r = \{E^r_{p,q}\}_{r\geq 1}$ with differential

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that $E^{r+1} \cong H_*(E^r)$.

A morphism $f : E \to E'$ of spectral sequences is a family of morphisms of complexes $f^r : E^r \to E'^r$ such that f^{r+1} is the morphism $H_*(f^r)$ induced by f^r.

Dually we have cohomological spectral sequence.

Definition 14.4

A ***(cohomological) spectral sequence*** $E = \{E_r, d_r\}$ consists of a sequence of $\mathbb{Z}$-bigraded R-modules $E_r = \{E_r^{p,q}\}_{r\geq 1}$ with differential

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that $E_{r+1} \cong H^*(E_r)$

Observe by setting $E_r^{p,q} = E^r_{-p,-q}$, those two are just the same.

From an exact couple

$$\begin{array}{ccc} A & \xrightarrow{i} & A \\ & {}_{k}\nwarrow \quad \swarrow_{j} & \\ & B & \end{array}$$

we can form a spectral sequence by repeatedly taking derived couples. That is, let $E^1 = (A, B)$, and let $E^r = (A^r, B^r)$ be after taking derived couple r times on E^1. Then $\{B^r, j^r \circ k^r\}$ forms a spectral sequence (you actually need to prove this!).

14.1 Bockstein Spectral Sequence

Let $C = C_\bullet$ be a torsion free chain complex over $\mathbb{Z}$. By tensor with C on the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ we get

$$0 \to C \to C \to C \otimes \mathbb{Z}/p\mathbb{Z} \to 0$$

Now take the induced homology sequence of this short exact sequence, we get

$$\begin{array}{ccc} H_*(C) & \longrightarrow & H_*(C) \\ & \nwarrow \qquad \swarrow & \\ & H_*(C \otimes \mathbb{Z}/p\mathbb{Z}) & \end{array}$$

and hence get an associated spectral sequence. This resulting sequence is called mod p Bockstein spectral sequence, with $d^r : E^r_n \to E^r_{n-1}$ for all $r \geq 1$, and we have short exact sequence

$$0 \to (p^{r-1}H_n(C)) \otimes \mathbb{Z}/p\mathbb{Z} \to E^r_n \to \mathrm{Tor}(p^{r-1}H_{n-1}(C), \mathbb{Z}/p\mathbb{Z}) \to 0$$

When $r = 1$, this is just the universal coefficient exact sequence.

We can describe this spectral sequence as follows. Let Σ^n be the functor on graded abelian groups given by $(\Sigma^n A)_{q+n} = A_q$. For a cyclic abelian group π, we have $\mathbb{Z}$-free resolution $C(\pi)$ given by $\mathbb{Z}$ in degree 0 if $\pi = \mathbb{Z}$ and by copies of $\mathbb{Z}$ in degrees 0 and 1 with differential $\times q^s$ if $\pi = \mathbb{Z}/q^s$. Assume $H_*(C)$ is of finite type and write $H_n(C)$ as direct sum of cyclic groups. For each cyclic summand, choose a representative cycle x and, if $\pi = \mathbb{Z}/q^s$, a chain y such that $d(y) = q^s x$. For each summand π, these choices determine a chain map $\Sigma^n C(\pi) \to C$. Summing over the cyclic summands and over n, we obtain a chain complex C' and a chain map $C' \to C$ that induces an isomorphism on homology and on Bockstein spectral sequences.

The Bockstein spectral sequences $\{E^r\}$ of $\Sigma^n C(\pi)$ are easy to compute. When $\pi = \mathbb{Z}$, $E^r_n = \mathbb{Z}$ and $E^r_m = 0$ for all $m \neq n$ and all r. When $\pi = \mathbb{Z}/q^s$ for $q \neq p$, $E^r_n = 0$ for all n and r. When $\pi = \mathbb{Z}/p^s$, $E^1 = E^s$ is $\mathbb{F}_p$ in degrees n and $n+1$, $d^s : E^s_{n+1} \to E^s_n$ is an isomorphism, and $E^r = 0$ for $r > s$. Returning to C, we see

$$E^\infty \cong (H_*(C)/TH_*(C)) \otimes \mathbb{F}_p$$

where $T(\pi)$ denotes the torsion subgroup of π. Moreover, there is one summand $\mathbb{Z}?p^s$ in $H_*(C)$ for each summand $\mathbb{F}_p$ in the vector space $d^s E^s$.

Remark 14.5

The homology group $H_*(C)$ can be computed by the mod p Bockstein spectral sequences where p range over all the primes p.

14.2 Spectral Sequence of Filtered Complex

Let K be a differential complex with differential operator D. A subcomplex K' of K is a subobject(e.g. submodule) such that $DK' \subseteq K'$. A sequence of subcomplexes

$$K = K_0 \supseteq K_1 \supsetneq K_2 \supsetneq K_3 \supsetneq \ldots$$

is called a filtration on K. This makes K into a filtered complex.

Remark 14.6

Normally we will also have a grading on K, i.e. $K = \bigoplus_{z\in\mathbb{Z}} C^k$, and we assume D has degree one, i.e. $D|_{C^k} : C^k \to C^{k+1}$. However, this is not necessary.

Definition 14.7

Let K be a ***filtered complex***, say $K = K_0 \supsetneq K_1 \supsetneq K_2 \ldots$, then we define its ***associated graded complex***

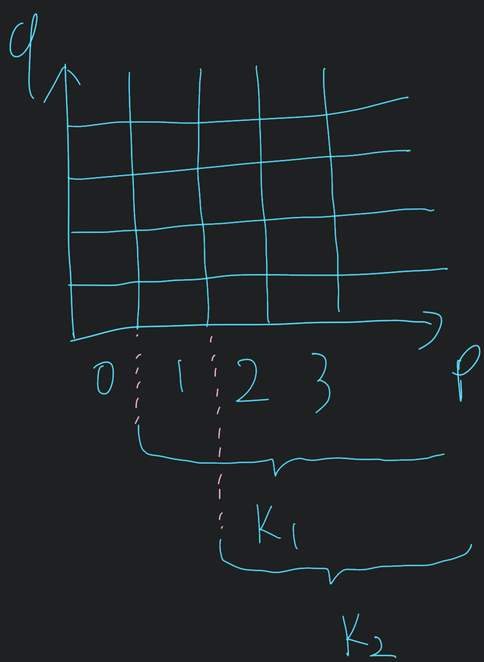
$$\mathrm{gr}(K) = \bigoplus_{p=0}^{\infty} K_p / K_{p+1}$$

For notational reason (since we are indexing stuff with $\mathbb{Z}$), we will extend the filtration to negative indices by setting $K_p = K$ if $p < 0$.

Example 14.8

Let $K = \bigoplus_{p,q} K^{p,q}$ be a double complex with horizontal operator δ and vertical operator d. Then we can form a single complex by setting $K = \bigoplus_k C^k$ where $C^k = \bigoplus_{p+q=k} K^{p,q}$. The differential $D : C^k \to C^{k+1}$ is defined by $D = \delta + (-1)^p d$. In this case, we get a filtration K_p by

$$K_p = \bigoplus_{i \geq p} \bigoplus_{q \geq 0} K^{i,q}$$



Now, suppose we have filtered complex K with differential D, we can define

$$A = \bigoplus_{p \in \mathbb{Z}} K_p$$

and thus its a differential complex with D. Now we can define $i : A \to A$ by the inclusion $K_{p+1} \hookrightarrow K_p$ and B be the cokernel of i, i.e. we get

$$0 \to A \xrightarrow{i} A \xrightarrow{j} B \to 0$$

where j is the projection to the quotient $A/\operatorname{im}(i)$. Its not hard to see

$$B = \operatorname{gr}(K) = \bigoplus_{p=0}^{\infty} K_p/K_{p+1}$$

Now, each of the term in the above short exact sequence $0 \to A \to A \to B \to 0$ is also a complex with differential induced from D.

Since we have a short exact sequence of complexes, we get induced long exact sequence on cohomology, i.e. we get

$$0 \to H^0(A) \xrightarrow{i_1} H^0(A) \xrightarrow{j_1} H^0(B) \xrightarrow{k_1} H^1(A) \xrightarrow{i_1} H^1(A) \xrightarrow{j_1} H^1(B) \xrightarrow{k_1} \dots$$

In other word, we get an exact couple

$$\begin{array}{ccc} H(A) & \xrightarrow{i_1} & H(A) \\ {\scriptstyle k_1}\nwarrow & & \swarrow{\scriptstyle j_1} \\ & H(B) & \end{array}$$

Let us set $A_1 = H(A)$ and $B_1 = H(B)$, and just write i instead of i_1 here, as we are going to repeat the above process over and over. Now, from this above exact couple

$$\begin{array}{ccc} A_1 & \xrightarrow{i} & A_1 \\ {\scriptstyle k_1}\nwarrow & & \swarrow{\scriptstyle j_1} \\ & B_1 & \end{array}$$

we can take its derived couple and get

$$\begin{array}{ccc} A_2 & \xrightarrow{i} & A_2 \\ {\scriptstyle k_2}\nwarrow & & \swarrow{\scriptstyle j_2} \\ & B_2 & \end{array}$$

and so on.

Example 14.9

Suppose our filtered complex terminates after K_3, i.e. we have

$$\dots = K_{-1} = K_0 \supsetneq K_1 \supsetneq K_2 \supsetneq K_3 \supsetneq 0$$

Thus we see

$$A = \dots \oplus K_0 \oplus K_0 \oplus K_1 \oplus K_2 \oplus K_3$$

and by taking cohomology we get

$$0 \to H(K_3) \to H(K_2) \to H(K_1) \to H(K) \to H(K) \to H(K)$$

and thus A_1 is the direct sum of all the terms above. Now A_2 is obtained by taking A_1 and apply i to it, i.e. A_2 is the direct sum of all the terms in the following sequence

$$0 \to iH(K_3) \to iH(K_2) \to iH(K_1) \to H(K) \to H(K) \to \dots$$

Similarly A_3 is obtained by apply two i to the sequence of A_1, and A_4 is the direct sum of terms in the sequence

$$0 \to iiiH(K_3) \to iiH(K_2) \to iH(K_1) \to H(K) \to H(K) \to \ldots$$

Now, since i pushes the $(p+1)$th pieces to pth pieces, and we only have 3 pieces, when applying i three times, we just get an inclusion $iiiH(K_3) \to iiH(K_2)$, and so on, i.e. all the arrows in the last sequence is just inclusion. This means A_n actually also equal A_4, for all $n \geq 4$. Now, since $i : A_4 \to A_4$ is inclusion and we have an exact couple, the map $k_4 : B_4 \to A_4$ must be the zero map, i.e.

$$B_4 = B_5 = B_6 = \ldots$$

Since the A_n and B_n becomes stationary, we will just denote this by A_∞ and B_∞, and we have exact couple

$$\begin{array}{ccc} A_\infty & \xrightarrow{i_\infty} & A_\infty \\ {\scriptstyle k_\infty}\nwarrow & & \swarrow{\scriptstyle j_\infty} \\ & B_\infty & \end{array}$$

and A_∞ is the direct sum of all the cohomology objects/groups

$$0 \subseteq iiiH(K_3) \subseteq iiH(K_2) \subseteq iH(K_1) \subseteq H(K) = H(K) = \ldots$$

In this case, B_∞ is just the associated graded complex of the filtered complex $H(K)$.

In general, suppose we have filtration of subcomplexes $K = K_0 \supsetneq K_1 \supsetneq K_2 \supsetneq \ldots$, then we get a sequence of cohomology objects

$$\ldots H(K_3) \xrightarrow{i} H(K_2) \xrightarrow{i} H(K_1) \xrightarrow{i} H(K) \to H(K) \to \ldots$$

Take F_p be the image of $H(K_p)$ in $H(K)$, we get a filtration of $H(K)$ by those F_p's, i.e.

$$H(K) = F_0 \supsetneq F_1 \supsetneq F_2 \supsetneq F_3 \supsetneq \ldots$$

We call this the induced filtration on $H(K)$.

If the filtration on K is bounded below, i.e. only finitely many non-zero terms going down (we say K has ***finite length***, and call the number l such that $K_l \neq 0$ and $0 = K_{l+1} = K_{l+2} = \ldots$ as the ***length***), then by the exact same argument as in the above example, we see A_r and B_r will be eventually stationary, and the value of B_∞ is just $\mathrm{gr}(H(K))$, where $H(K)$ is filtered by the F_p.

In the above, we often write E_r instead of B_r, and the differentials are given by $d_r = j_r \circ k_r$, where we see $E_r = H(E_{r-1})$.

Definition 14.10

Let $\{E_r, d_r\}$ be a spectral sequence with E_∞ exists. If E_∞ is equal to the associated graded group of some filtered group H, then we say the spectral sequence

converges to H.

Now we suppose K comes with a grading $K = \bigoplus_{n\in\mathbb{Z}} K^n$, and to distinguish the grading degree n from the filtration degree p, we call n the dimension of K^n. The filtration $\{K_p\}$ on K induces a filtration in each dimension, i.e. denote $K_p^n := K^n \cap K_p$, then $\{K_p^n\}$ is a filtration on K^n.

Theorem 14.11

Let $K = \bigoplus_{n\in\mathbb{Z}} K^n$ be a graded filtered complex with filtration $\{K_p\}$ and $H_D^*(K)$ the cohomology of K with filtration given by the image of $H(K_p)$ in $H(K)$. Suppose for each dimension n the filtration $\{K_p^n\}$ has finite length (i.e. only finitely many non-zero terms going down). Then the short exact sequence

$$0 \to \bigoplus_p K_{p+1} \to \bigoplus_p K_p \to \mathrm{gr}(K) \to 0$$

induces a spectral sequence which converges to $H_D^*(K)$.

Proof. By treating the convergence question one dimension at a time, this proof reduces to the ungraded situation. Indeed, as before,

$$A_r = \bigoplus_{p\in\mathbb{Z}} i^{r-1} H(K_p)$$

if $r \geq p+1$, then $i^r H(K_p) = F_p$ and

$$i : i^r H(K_{p+1}) \to i^r H(K_p)$$

is an inclusion. With a grading on each derived couple, i and j preserve the dimension, but k increases the dimension by 1. Given n, let $\ell(n)$ be the length of $\{K_p^n\}$ and let $r \geq \ell(n+1)+1$. Then for any integer p,

$$i^r H^{n+1}(K_{p+1}) = F_{p+1}^{n+1}$$

and

$$i : i^r H^{n+1}(K_{p+1}) \to i^r H^{n+1}(K_p)$$

is an inclusion. Thus

$$i_r : A_r^{n+1} \to A_r^{n+1}$$

is an inclusion and

$$k_r : B_r^n \to A_r^{n+1}$$

is the zero map. Therefore, as $r \to \infty$, the group B_r^n becomes stationary and we can define B_∞^n as this stationary value. Note

$$A_\infty^n = \bigoplus_p F_p^n$$

and that i_∞ sends F_{p+1}^n into F_p^n for every n. Because $i_\infty : \bigoplus_p F_{p+1} \to \bigoplus_p F_p$ is an inclusion, B_∞ is the associated graded complex $\bigoplus_p F_p/F_{p+1}$ of $H_D^*(K)$.

14.3 Spectral Sequence of Double Complex

Now take $K = \bigoplus K^{p,q}$ be double complex with a filtration $\{K_p\}$ as in Example 14.8. In this subsection we will focus on this particular case and obtain refined version of Theorem 14.11.

Observe that $A = \bigoplus K_p$ is also a double complex, and we can form a single complex $A = \bigoplus A^k$ by summing the bidegrees, i.e. A^k consists of all elements of A whose total degree is k. We get inclusion $i : A^k \to A^k$ by

$$i : A^k \cap K_{p+1} \to A^k \cap K_p$$

The single complex A inherits the differential operator $D = \delta + (-1)^p d$ from K. Similarly, $B = \bigoplus K_p/K_{p+1}$ can be made into a single complex with opeartor D. Note the differential operator D on B is $(-1)^p d$. Thus

$$E_1 = H_D(B) = H_d(K) \tag{Eq. 14.1}$$

Recall the coboundary operator $k_1 : H(B) \to H(A)$ is the coboundary operator of the short exact sequence $0 \to A \xrightarrow{i} A \xrightarrow{j} B \to 0$ and hence is defined by the following diagram

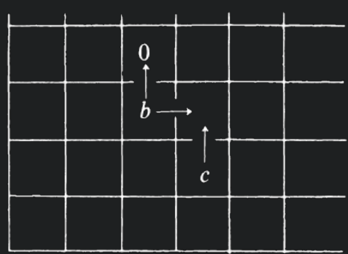
$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A^{k+1} \cap K_{p+1} & \xrightarrow{(3)} & A^{k+1} \cap K_p & \longrightarrow & B^{k+1} \cap K_p/K_{p+1} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle D} & & {\scriptstyle (2)}\uparrow & & \uparrow{\scriptstyle D} & & \\
0 & \longrightarrow & A^{k} \cap K_{p+1} & \longrightarrow & A^{k} \cap K_p & \xrightarrow[(1)]{} & B^{k} \cap K_p/K_{p+1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array} \tag{Eq. 14.2}$$

Let b in $A^k \cap K_p$ represent a cocycle $[b]$ in $B^k \cap K_p/K_{p+1}$. This correspond to the (1) in the above. To get $k_1([b])$ we need to go through (2) and (3) above, which correspond to compute Db and take its inverse under i.

Since b represents an element of $E_1 = H_D(B) = H_d(K)$ such that $db = 0$, we see $Db = \delta b + (-1)^p db = \delta b$. Thus, $k_1[b] = [\delta b]$, and so the differential $d_1 = j_1 k_1$ on E_1 is given by δ. Consequently,

$$E_2 = H_\delta H_d(K) \tag{Eq. 14.3}$$

We now compute d_2 on E_2. Observe an element of $E_2 = H_\delta H_d(K)$ is represented by an element b in K such that $db = 0$ and $\delta b = -D''c$ for some $c \in K$, where $D'' = (-1)^p d$, i.e. we have



We will denote the class of b in E_r, if it is defined, by $[b]_r$. From the definition of derived couples we see

$$d_2[b]_2 = j_2k_2[b]_2 = j_2k_1[b]_1$$

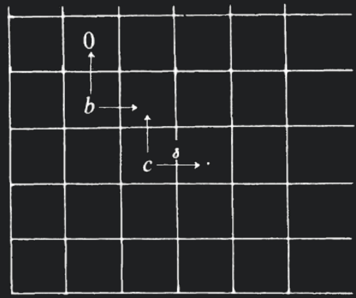
To compute $j_2k_1[b]_1$, we need to find a such that $k_1[b]_1 = i[a]_1$. Then $j_2k_2[b]_2 = [j_1a]_2$. Since k_1b is in $A^{k+1} \cap K_{p+1}$, a lies in $A^{k+1} \cap K_{p+2}$. To find a we use not b but $b+c$ in $A^k \cap K_p$ to represent $[b]_2$ in (1) of the diagram Eq. 14.2. This is possible since b and $b+c$ have the same image under the projection $K_p \to K_p/K_{p+1}$. Then

$$k_1(b+c) = D(b+c) = \delta c$$

and thus

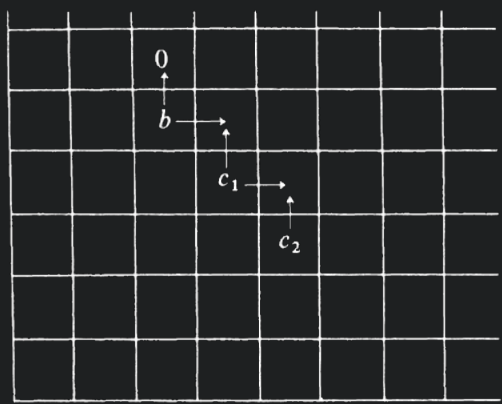
$$d_2[b]_2 = [\delta c]_2 \qquad \text{(Eq. 14.4)}$$

Thus the differential d_2 is given by the δ of the tail of the zig-zag which extends b, i.e.



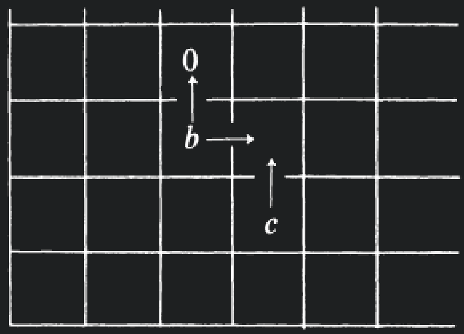
It is easy to show δc represents an element of $H_\delta H_d(K)$ and that the definition of $d_2[b]_2$ is independent of the choice of c.

Now, observe that if $d_2[b]_2 = 0$, then we can find c_1, c_2 such that b can be extended to a zig-zag as shown

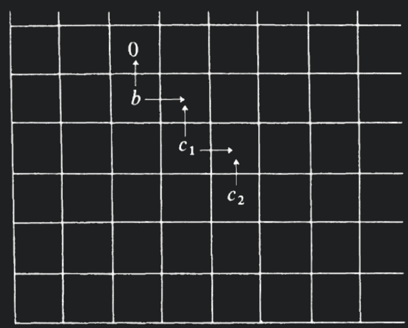


where $D''b = 0$, $\delta b = -D''c_1$ and $\delta c_1 = -D''c_2$.

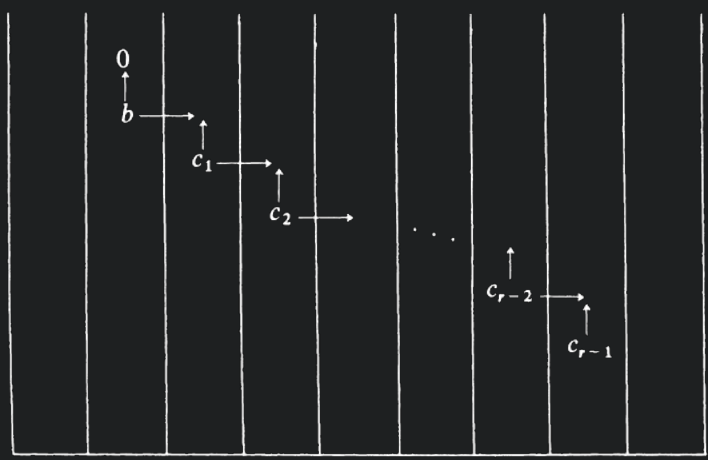
We say that an element b in K ***lives*** to E_r if it represents a cohomology class in E_r, i.e. b is a cocycle in $E_1, E_2, \ldots, E_{r-1}$. From the discussion above, we see b lives to E_2 if it can be extended to a zig-zag of length 2, where the length being the number of terms in it, i.e.



with $db = 0$, $\delta b = -D''c$ and $d_2[b]_2 = [\delta c]_2$. It lives to E_3 if it can be extended to a zig-zag of length 3, i.e.



with $db = 0$, $\delta b = -D''c_1$ and $dc_1 = -D''c_2$. To compute $d_3[b]_3$, we use $b + c_1 + c_2$ in $A^k \cap K_p$ to represent $[b] \in B^k \cap (K_p/K_{p+1})$ in (1) of Eq. 14.2, so $k_3[b]_3$ is given by $D(b + c_1 + c_2) = \delta c_2$ and $d_3[b]_3 = [\delta c_2]_3$. In general, parallel to the discussion above, an element $b \in K^{p,q}$ lives to E_r if it can be extended to a zig-zag of length r, i.e.



and the differential d_r on E_r is given by δ of the tail of the zig-zag

$$d_r[b]_r = [\delta c_{r-1}] \qquad \text{(Eq. 14.5)}$$

Thus, we see the bidegrees (p, q) of the double complex $K = \bigoplus K^{p,q}$ persist in the spectral sequence

$$E_r = \bigoplus_{p,q} E_r^{p,q}$$

and d_r shifts the bidegrees by $(r, -r+1)$:

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

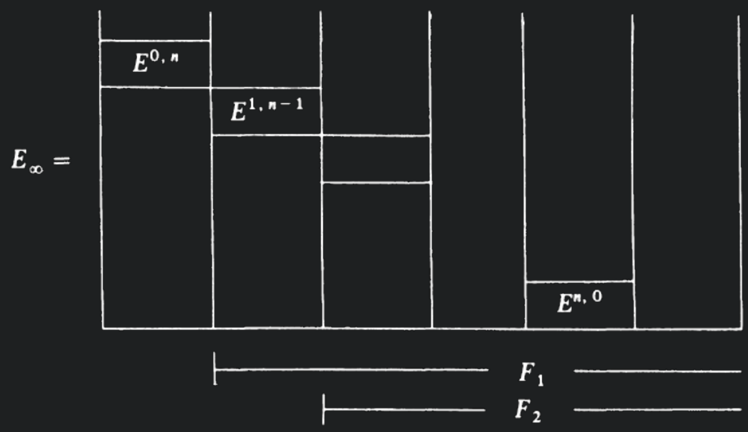
The filtration on $H(K) = \bigoplus H^n(K)$:

$$H(K) = F_0 \supsetneq F_1 \supsetneq F_2 \supsetneq F_3 \supsetneq F_4 \dots$$

induces a filtration on each component $H^n(K)$, the successive quotients of the filtration being $E_\infty^{0,n}$, $E_\infty^{1,n-1}$,...,$E_\infty^{n,0}$:

$$H^n(K) = (F_0 \cap H^n) \supsetneq (F_1 \cap H^n) \supseteq (F_2 \cap H^n) \supsetneq \dots \supsetneq (F_n \cap H^n) \supsetneq 0 \qquad \text{(Eq. 14.6)}$$

with $E_\infty^{0,n} = (F_0 \cap H^n)/(F_1 \cap H^n)$, $E_\infty^{1,n-1} = (F_1 \cap H^n)/(F_2 \cap H^n)$ and so on. In other word, we have



In short, we have proved the following result:

Theorem 14.12

Given a double complex $K = \bigoplus_{p,q \geq 0} K^{p,q}$, there is a spectral sequence $\{E_r, d_r\}$ converging to the total cohomology $H_D(K)$ such that each E_r has a bigrading with

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

and

$$E_1^{p,q} = H_d^{p,q}(K), \quad E_2^{p,q} = H_\delta^{p,q} H_d(K)$$

Furthermore, the associated graded complex of the total cohomology is given by

$$\mathrm{gr}(H_D^n(K)) = \bigoplus_{p+q=n} E_\infty^{p,q}(K)$$

Now, observe instead of using the filtration in Example 14.8, we can define a filtration on K by

$$K_q = \bigoplus_{j \geq q} \bigoplus_{p \geq 0} K^{p,j}$$

This gives spectral sequence $\{E'_r, d'_r\}$ also converging to the total cohomology $H_D(K)$, but with

$$E'_1 = H_\delta(K), \quad E'_2 = H_d H_\delta(K)$$

and

$$d'_r : E'^{p,q}_r \longrightarrow E'^{p-r+1,q+r}_r$$

Example 14.13

Let M be a manifold with good cover $\mathcal{U}$. Consider double complex $K = \bigoplus K^{p,q}$ with

$$K^{p,q} = \prod_{a_0<\ldots<a_p} \Omega^q(U_{a_0} \cap \ldots \cap U_{a_p}) =: C^p(\mathcal{U}, \Omega^q)$$

where $\Omega^q(X)$ is the q-forms on X. Then, the rows of K are the Mayer-Vietoris sequences (e.g. if $M = U \cup V$ is a good cover then its Mayer-Vietoris sequence is $0 \to \Omega^*(U \cup V) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0$). Thus, the E_1 term of the second spectral sequence (i.e. we use the filtration $K_q = \bigoplus_{j\geq q} \bigoplus_{p\geq 0} K^{p,j}$) $E'_1 = D_\delta$ is given by

$\Omega^3(M)$	0		
$\Omega^2(M)$	0		
$\Omega^1(M)$	0		
$\Omega^0(M)$	0		

Thus, E_2 is taking H on E_1, and by definition this is the de Rham cohomology, i.e. $E'_2 = H_d H_\delta$ is equal

$H^3_{DR}(M)$	0		
$H^2_{DR}(M)$	0		
$H^1_{DR}(M)$	0		
$H^0_{DR}(M)$	0		

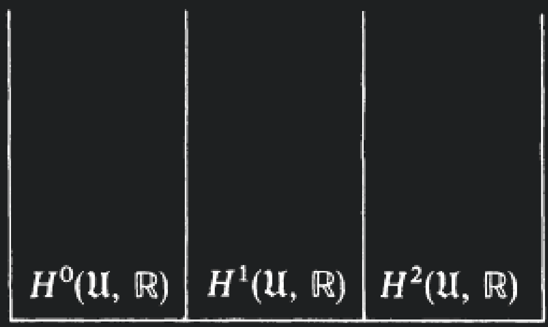
In general, a spectral sequence is said to be degenerate at E_r if $d_r = d_{r+1} = \ldots = 0$. For such a spectral sequence, $E_r = E_{r+1} = E_{r+2} = \ldots = E_\infty$. The degeneration of the second spectral sequence $\{E'_r, d'_r\}$ at E_2 concludes that we have an isomorphism

$$H^k_{dR}(M) = \bigoplus_{p+q=k} H^{p,q}_D\{C^*(\mathcal{U}, \Omega^*)\}$$

Now consider the spectral sequence we obtained by the filtration of Example 14.8. Its E_1 term this time is

$$E_q^{p,q} = \prod_{a_0<\ldots<a_p} H^q(U_{a_0} \cap \ldots \cap H_{a_p}) = \begin{cases} 0 & \text{if } q > 0 \\ C^p(\mathcal{U}, \mathbb{R}) & \text{if } q = 0 \end{cases}$$

where $C^p(\mathcal{U}, \mathbb{R})$ consists of locally constant functions on $(p+1)$-fold intersections $U_{a_0} \cap \ldots \cap U_{a_p}$. By taking the E_2 page, we are taking the cohomology on the cochain complex $C^p(\mathcal{U}, \mathbb{R})$, which is just the Čech cohomology. That is, $E_2 = H_\delta H_d$ is given by



The degeneration of this spectral sequence gives

$$H^k(\mathcal{U}, \mathbb{R}) = \bigoplus_{p+q=k} E_2^{p,q} = \bigoplus_{p+q=k} E_\infty^{p,q} = H_D^k(C^*(\mathcal{U}, \Omega^*))$$

Together we concluded

$$H_{dR}^k(M) = H^k(\mathcal{U}, \mathbb{R}) \quad \text{for all integers } k$$

In particular, what this means is that the de Rham cohomology is isomorphic to the Čech cohomology, and the Čech cohomology does not depend on the choice of good cover (as its isomorphic to $H_{dR}^k(M)$).