Contents

The goal of this note is to go through the book “height in dio geo”, chapter 10, 11 and 14. In particular,

1. chapter 10 is based on chapter 8 and 9.
2. chapter 11 is based on chap $2, 8, 9, 10$.
3. chapter 14 is based on chap 12 (abc conjecture), chap 13 (Nevanlinna theory).

Hence, we organize the study into a brief introduction to the naive height theory, without going as deep as the subspace theorem. Then, we immediately begin study chapter 8 and 9, and then proceed to the three main chapters I want to cover.

Chapter 1

Heights

Throughout the book, it is safe to assume we are working with number fields only (so no function fields).

In particular, this chapter is more detailed than necessary for our purpose, just so that we start slowly.

1.1 A Bit Algebraic Number Theory

Definition 1.1.1

A ***place*** v is an equivalence class of non-trivial absolute value on K, where two absolute values $v \sim v'$ if they induce the same topology.

Remark 1.1.2

Recall two absolute values $|\cdot|_1$ and $|\cdot|_2$ are equivalent if and only if there is real number $s > 0$ so $|x|_1 = |x|_2^s$ for all $x \in K$.

Let L/K be a field extension, and w be a place on L and v a place on K, then we write $w \mid v$ to mean $v|_K = w$, or more precisely, any representative of w restrict to K is a representative of v.

Definition 1.1.3

The completion of K with respect to the place v is an extension field K_v with place w of K, such that:

1. $w \mid v$
2. The topology of K_v induced by w is complete
3. K is dense subset of K_v in the above topology

Let $K = \mathbb{Q}$, then the ordinary absolute value $|\cdot| := |\cdot|_\infty$ gives $\mathbb{R}$ as its completion. On the other hand, for prime number p define $|m/n|_p := p^{-a}$, where a is the unique number such that $m/n = p^a \cdot (m'/n')$ with $\gcd(m',p) = 1 = \gcd(n',p)$. Equivalently, $|\cdot|_p$ is uniquely determined by the condition

$$|q|_p := \begin{cases} 1 & \text{for primes } q \neq p \\ \frac{1}{p} & \text{if } p = q \end{cases}$$

The completion of this is the p-adic numbers and we denote by $\mathbb{Q}_p$.

Recall we call an absolute value non-archimedean if $|x+y| \leq \max(|x|,|y|)$ for all $x,y \in K$. Thus, if $|x+y| < \max(|x|,|y|)$ for some $x,y \in K$ then we call this absolute value archimedean.

Theorem 1.1.4

The only complete archimedean fields are $\mathbb{R}$ and $\mathbb{C}$.

Recall that for fintie extension L/K, we define the norm $N_{L/K}$ and trace $T_{L/K}$ as follows. Each $a \in L$ determines a K-linear map $m : L \to L$ by $x \mapsto ax$, and we define

$$N_{L/K}(a) = \det(m_a), \quad T_{L/K}(a) := \operatorname{tr}(m_a)$$

Example 1.1.5

If L/K is Galois extension, then

$$N_{L/K}(a) = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(a)$$

Explicitly, if $L = \mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$, then $N(a+b\sqrt{2}) = (a+b\sqrt{2})(a-b\sqrt{2})$ because the Galois group in this case has order 2, and its generated by the element which sends $\sqrt{2}$ to $-\sqrt{2}$.

Not Relevant

More generally, for $f : X \to Y$ finite locally free morphism of schemes of rank $k > 0$, we can define a norm $N_{X/Y} : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ as follows. By assumption, $f_*\mathcal{O}_X$ is finite locally free $\mathcal{O}_Y$-algebra, and thus we can define a morphism of sheaves $N_{f_*\mathcal{O}_X/\mathcal{O}_Y} : f_*\mathcal{O}_X \to \mathcal{O}_Y$ by $N_{f_*\mathcal{O}_X/\mathcal{O}_Y}(V)(b) := \det(m_b)$, where for $b \in \Gamma(V, f_*\mathcal{O}_X)$ we define $m_b : \Gamma(V, f_*\mathcal{O}_X) \to \Gamma(V, f_*\mathcal{O}_X)$ as the multplication by b.

Then for line bundle $\mathscr{L}$ on X, we see $f_*\mathscr{L}$ is an invertible $f_*\mathcal{O}_X$-module and thus we can find open cover $V = (V_i)$ of Y so $f_*\mathscr{L}$ is given by Čech 1-cocycle (g_{ij}) of $(f_*\mathcal{O}_X)^\times$, i.e. $g_{ij} \in \Gamma(V_i \cap V_j, (f_*\mathcal{O}_X)^\times)$ and $g_{kj}g_{ji} = g_{ki}$ on the triple intersection. Then one checks $(N_{f_*\mathcal{O}_X/\mathcal{O}_Y}(g_{ij}))$ is a Čech 1-cocycle of $\mathcal{O}_Y^\times$, i.e. it defines a line bundle on Y. This is the global norm map.

Proposition 1.1.6

Let K be a field which is complete with respect to place v and L/K finite extension.

Then there is a unique extension w of $|\cdot|_v$ on L, such that

$$|x|_w := |N_{L/K}(x)|_v^{1/[L:K]}$$

In particular, L is complete with respect to $|\cdot|_w$.

For K with non-archimedean place v and L a finite extension of K, define

$$R_v := \{x \in K : |x|_v \leq 1\}$$

This is a local ring with unique maximal ideal $\mathfrak{m}_v := \{x \in K : |x|_v = 1\}$. In particular, we have residue field $\kappa(v) := R_v/\mathfrak{m}_v$.

Definition 1.1.7

Let L/K be a finite extension and v a non-archimedean place on K and w extends v. Then we define:

1. the ***residue degree*** $f_{w/v}$ of L/K in w is the dimension of $\kappa(w)$ over $\kappa(v)$.
2. the ***ramification index*** $e_{w/v}$ of L/K is defined to be the index of the subgroup $|K^\times|_v$ in $|L^\times|_w$.

A place v is called ***discrete*** if $|K^\times|_v$ is cyclic. In this case, $\mathfrak{m}_v$ is a principal ideal and any generator is called a uniformizer.

Lemma 1.1.8: Hensel Lemma

Let K be a complete non-archimedean field with place v. Let $f \in K[t]$ be monic with reduction $\overline{f}(t) = \overline{g}(t)\overline{h}(t)$ in $\kappa(v)[t]$, where $\overline{g}$ and $\overline{h}$ are monic and coprime. Then there are monic $G, H \in R_v[t]$ with $f(t) = G(t)H(t)$ and $\overline{G}(t) = \overline{g}(t)$ and $\overline{H}(t) = \overline{h}(t)$.

Theorem 1.1.9: Approximation Theorem

Let $v_1, \ldots, v_n$ be inequivalent non-trivial absolute values on a field K. Then for $x_1, \ldots, x_n \in K$ and $\epsilon > 0$ there is $x \in K$ so

$$|x - x_k|_{v_k} < \epsilon$$

for $k = 1, \ldots, n$.

The next result classifies absolute values on finite extension L/K extending place v on K.

Proposition 1.1.10

Let L be a fintie extension of K and K is generated by a single element ξ. Let $f(t)$ be the monic minimal polynomial of ξ and $f(t) = f_1^{k_1}(t) \ldots f_r^{k_r}(t)$ be the decomposition into different irreducible monic factors $f_j(t) \in K_v[t]$. Then:

1. *for each $1 \leq j \leq r$ there is an injective morphism*

$$\iota : L \to K_j := K_v[t]/(f_j(t))$$

of field extensions over K, given by $\xi \mapsto t$, so that K_j is the completion of L with respect to $|\cdot|_j$ and ι
2. *there is a unique extension $|\cdot|_j$ of K_v to K_j, and they are pairwise inequivalent*
3. *for any absolute value $|\cdot|_w$ extending $|\cdot|_v$ to L, there is unique $1 \leq j \leq r$ so $|\cdot|_j$ on K_j restrict to L is $|\cdot|_w$*

Corollary 1.1.10.1

If L is finite separable extension of K, then

$$\sum_{w|v}[L_w : K_v] = [L : K]$$

where w is sum over all palces w of L with $w \mid v$.

In particular, we call the number $[L_w : K_v]$ the local degree of L/K in w.

Corollary 1.1.10.2

Let L/K be finite Galois extension with $G = \mathrm{Gal}(L/K)$, and w_0, w two absolute values on L extending v on K. Then there is $\sigma \in G$ such that

$$|x|_w = |\sigma(x)|_{w_0}$$

for all $x \in L$. The completions L_w and L_{w_0} are isomorphic over K_v (but need not be isomorphic over L).

For K with non-trivial absolute value w, and L/K with $w \mid v$, we define

$$\|x\|_w = |N_{L_w/K_v}(x)|_v$$

for $x \in L$ and

$$|x|_w := |N_{L_w/K_v}(x)|_v^{1/[L:K]}$$

By Proposition 1.1.6 we know the restriction of $|N_{L_w/K_v}(x)|_v^{1/[L:K]}$ to L is a representative of w extending v. This absolute value is the normalization of v.

Lemma 1.1.11

Let $x \in K\backslash\{0\}$ and $y \in L\backslash\{0\}$. Then

$$\sum_{w|v} \log |x|_w = \log |x|_v$$

$$\sum_{w|v} \log \|y\|_w = \log |N_{L/K}(y)|_v$$

Next we talk about the product formula.

Let K be a field, M_K a set of non-trivial places such that the set

$$\{|\cdot|_v \in M_K : |x|_v \neq 1\}$$

is finite for any $x \in K \backslash \{0\}$. Then we say M_K satisfies the ***product formula*** if

$$\prod_{v \in M_K} |x|_v = 1$$

for all $x \in K \backslash \{0\}$.

Now suppose M_K satisfies product formula, and let M_L be the set of places on L defined by the normalizations, i.e. $M_L = \{|N_{L_w/K_v}(\cdot)|_v^{1/[L:K]} : v \in M_K, w \mid v\}$.

Proposition 1.1.12

The set of places M_L as above also satisfies product formula, if M_K does.

Now, for $\mathbb{Q}$ we define

$$M_{\mathbb{Q}} = \{|\cdot|_p : p \text{ a prime or } p = \infty\}$$

where we take the usual representatives, i.e. $|p|_p = 1/p$ for p a prime, or the usual absolute value when $p = \infty$. Then, for any number field K, we define M_K as the set of places and normalized absolute values, obtained by the above construction to the extension $K/\mathbb{Q}$. In other words, for any number field K, we always define

$$M_K = \{|N_{K_w/\mathbb{Q}_p}(\cdot)|_p^{1/[K:\mathbb{Q}]} : p \in M_{\mathbb{Q}}, w \mid p\}$$

Proposition 1.1.13

If K be a number field, then M_K (defined as above) satisfies the product formula.

The proof of this can be reduced to the fact every integer can be factored uniquely into product of prime numbers.

Convention

In this note, whenever we talk about M_K for a number field K, it will always be the the of places over $M_{\mathbb{Q}}$ defined as above. In particular, for $M_{\mathbb{Q}}$ we will always use the normalized absolute values, i.e. $|p|_p = 1/p$ for all primes and $|x|_\infty$ the usual absolute value. Specifically, M_K consists of places v so that $v \mid p$ and

$$|x|_v = |N_{K_v/\mathbb{Q}_p}(x)|_p^{1/[K:\mathbb{Q}]}$$

for $x \in K$.

By the product formula, we obtain a refinement of the approximation theorem for number fields.

Theorem 1.1.14

Let $(|\cdot|_v)_{v\in S}$ be representatives for a fintie set S of non-archimedean places of number field K, $x_v \in K_v$ for every $v \in S$, and let $\epsilon > 0$. Then there is $x \in K$ with $|x - x_v|_v < \epsilon$ for all $v \in S$ and $|x|_v \leq 1$ for all non-archimedean $v \notin S$.

We will spend the remaining of this section investigate M_K for number field K more closely.

Given number field K of degree n, we know M_K consists of places $v \mid \infty$, and $v \mid p$ for some prime number p.

First assume v extends ∞. In this case, observe K_v must be either $\mathbb{R}$ or $\mathbb{C}$, as $\mathbb{Q}_\infty = \mathbb{R}$. By field theory we konw there are n many embeddings $\sigma : K \hookrightarrow \mathbb{C}$, and we see each can define an absolute value by $|x|_\sigma := |\sigma(x)|_\infty$, where $|\cdot|_\infty$ is the usual absolute value on $\mathbb{R}$ or $\mathbb{C}$, depends on $\mathrm{im}(\sigma)$ lies in $\mathbb{C}$ or $\mathbb{R}$. In particular, we see if $\mathrm{im}(\sigma)$ is not real, then σ and the conjugate $\overline{\sigma}$ defines the same absolute value. On the other hand, if $\mathrm{im}(\sigma) \subseteq \mathbb{R}$ then it gives one place. Thus, we see if (r_1, r_2) is the signature of K (i.e. r_1 is the number of real embeddings and $2r_2$ the number of complex embeddings), then we have $r_1 + r_2$ many distinct places in M_K extending $\infty \in M_{\mathbb{Q}}$.

Next, let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$, the ring of integers of K. Then $\mathfrak{p}$ lies over some prime number p. Then, we can define a valuation on $\mathcal{O}_K$ via $\mathrm{ord}_{\mathfrak{p}}(x)$ be the exponent of $\mathfrak{p}$ in the factorization of the fractional ideal xR_k. This extends to a map $\mathrm{ord}_{\mathfrak{p}} : K^\times \to \mathbb{Z}$, and thus we obtain a place associated to $\mathfrak{p}$. The normalization here is given by

$$|x|_{\mathfrak{p}} = p^{-\mathrm{ord}_{\mathfrak{p}}(x)/e_{\mathfrak{p}}}$$

where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$ over $\mathbb{Q}$.

In particular, we can prove those are all the places in M_K, i.e. M_K consists of two parts, one obtained by just computing all embeddings $K \hookrightarrow \mathbb{C}$, and one obtained by computing all primes in $\mathcal{O}_K$ lying over p, as p range over all primes of $\mathbb{Z}$.

1.2 Heights In Projective and Affine Spaces

Let $\overline{\mathbb{Q}}$ be a choice of algebraic closure of $\mathbb{Q}$, and $\mathbb{P}^n = \mathbb{P}^n_{\overline{\mathbb{Q}}}$ the projective space with global coordinates $\mathbf{x} = (x_0 : x_1 : \ldots : x_n)$. Let $P \in \mathbb{P}^n$, we will now define a function, called height, on algebraic points of $\mathbb{P}^n_{\overline{\mathbb{Q}}}$. This should be thought as a measure of the algebraic complication needed to describe the point P.

Let $P \in \mathbb{P}^n$ be represented by homogeneous coordinate $(P_0 : \ldots : P_n)$, where $P_0, \ldots, P_n \in K$ for some number field K. Then we define

$$h(P) := \sum_{v \in M_K} \max_j \log |P_j|_v$$

Lemma 1.2.1

$h(P)$ is independent of the choice of K.

Proof. Let L be another number field containing the coordinates $P_0, \dots, P_n$ of P. We can assume $K \subseteq L$. Then

$$\sum_{w \in M_L} \max_j \log |P_j|_w = \sum_{v \in M_K} \sum_{w|v} \max_j \log |P_j|_w$$

Now by Lemma 1.1.11 we see $\sum_{w|v} \log |x|_w = \log |x|_v$ for any $x \in K \setminus \{0\} \subseteq L \setminus \{0\}$, which concludes our proof.

Lemma 1.2.2

$h(P)$ is independent of the choice of coordinates.

Proof. Let Q be another coordinate representing the same point of $\mathbb{P}^n_{\overline{\mathbb{Q}}}$. By the above, we may assume $Q, P \in \mathbb{P}^n_K$ for number field K. Thus, there is $\lambda \in K \setminus \{0\}$ so $Q = \lambda P$. Thus

$$h(Q) = \sum_{v \in M_K} \log |\lambda|_v + \sum_{v \in M_K} \max_j \log |P_j|_v$$

where $\sum_{v \in M_K} \log |\lambda|_v = 0$ by product formula, and we are done.

Definition 1.2.3

We call $h(P)$ the ***absolute log height*** (briefly, ***height***) of P. We also define the multiplicative height $H(P) := e^{h(P)}$.

Example 1.2.4

Let α be an algebraic integer in a number field K of degree n.

We can identify α as the point $(\alpha : 1)$ in $\mathbb{P}^1_K$, and compute its height. In particular, we see

$$h(\alpha) = \sum_{v \in M_K} \log(\max(|\alpha|_v, 1))$$

Then note $\alpha \mathcal{O}_K$ factors as a bunch of prime ideals of $\mathcal{O}_K$ with all exponents, and thus almost all $|\alpha|_\mathfrak{p}$ should be less than 1, except one of them equal 1 (here we are using the fact α lies in $\mathcal{O}_K$). Hence, we see

$$h(\alpha) = \sum_{v|\infty} \log(\max(|\alpha|_v, 1))$$

For example, if we take $\alpha = i$, then we have two embeddings of $\mathbb{C} \hookrightarrow \mathbb{C}$, the trivial one and the conjugate. Hence

$$h(i) = \log(\max(|i|_\infty, 1)) + \log(\max(|-i|_\infty, 1)) = 0$$

Similarly, if we take $\sqrt{2}+1 \in \mathbb{Q}[\sqrt{2}]$, then we have two embeddings and so

$$\begin{aligned} h(\sqrt{2}+1) &= \log(\max(|1+\sqrt{2}|_\infty, 1)) + \log(\max(|1-\sqrt{2}|_\infty, 1)) \\ &= \frac{1}{2}\log(1+\sqrt{2}) \end{aligned}$$

More generally, if $\alpha \in K$ is an algebraic number, and write $\alpha\mathcal{O}_K = \mathfrak{b}/\mathfrak{c}$ for relative prime ideals of $\mathcal{O}_K$. Then

$$h(\alpha) = N(\mathfrak{b}) + \sum_{v|\infty} \log(\max(|\alpha|_v, 1))$$

where $N(\mathfrak{b})$ is the absolute norm of the ideal $\mathfrak{b}$.

Remark 1.2.5

Let $S \subseteq M_K$ be a finite set of places, which includes the set S_∞ of all archimedean places of K. Then we say $x \in K$ is an S-integer if $|x|_v \leq 1$ for all $v \notin S$. The S-integers of K form a subring $\mathcal{O}_{S,K}$ of K. The units in $\mathcal{O}_{S,K}$ are called the S-units of K, and form a group $\mathcal{U}_{S,K}$. An element $x \in \mathcal{O}_{S,K}$ is S-unit if and only if $|x|_v = 1$ for all $v \notin S$.

In particular, we can show S_∞-integers is the same as an algebraic integer. Indeed, x is S_∞-integer, then $|x|_v \leq 1$ for all non-archimedean places, i.e. $x\mathcal{O}_K$ decomposes as a bunch of primes with only positive exponents, i.e. $x \in \mathcal{O}_K$.

Theorem 1.2.6: Kronecker

The height of $\xi \in \overline{\mathbb{Q}}^\times$ is zero if and only if ξ is a root of unity.

Proof. Let K be a number field and $\xi \in K^\times$. If ξ is a root of unity, then its absolute values are all equal 1 and hence its height is 0.

Conversely, suppose $h(\xi) = 0$, then $|\xi|_v \leq 1$ for every $v \in M_K$, i.e. ξ is algebraic integer. Let d be the degree of ξ and $\mathbf{x} = (\xi_1, \ldots, \xi_d)$ a full set of conjugates of ξ. Then, consider, for every integer $m > 0$, the elementary symmetric functions $s_i(\xi_1^m, \xi_2^m, \ldots, \xi_d^m)$, for $i = 0, \ldots, d$. Since ξ is an algebraic integer, we see $s_i(\xi_1^m, \ldots, \xi_d^m) \in$

$\mathbb{Z}$, as we have equality

$$(x-\xi_1^m)\ldots(x-\xi_d^m)=\sum_{i=0}^{d}(-1)^i s_i(\xi_1^m,\ldots,\xi_d^m)x^d$$

where the LHS lies in $\mathbb{Z}[x]$ as its the minimal polynomial of ξ^m.

Since $|\xi_j|_v \leq 1$ for every j and v and since $s_i(\xi_1^m,\ldots,\xi_d^m)$ is the sum of $\binom{d}{i}$ terms each of which is a product of factors not excedding 1 in absolute value, its not clear

$$\sum_{i=0}^{d}|s_i(\xi_1^m,\ldots,\xi_d^m)| \leq \sum_{i=0}^{d}\binom{d}{i}=2^d$$

This means that for the set $\{(\xi_1^m,\ldots,\xi_d^m) : m \geq 1\}$ there are only finitely many possible values for $\sum_i |s_i(\xi^m)|$. Hence there must be two distinct integers m and n, so

$$\sum_{i=0}^{d}|s_i(\xi_1^m,\ldots,\xi_d^m)|=\sum_{i=0}^{d}|s_i(\xi_1^n,\ldots,\xi_d^n)|$$

This happens iff $(\xi_1^m,\ldots,\xi_d^m)=(\xi_{\sigma(1)}^n,\ldots,\xi_{\sigma(d)}^n)$ for some permutation σ. Now repeat this argument $\mathrm{ord}(\sigma)$ many times we can assume $\sigma=\mathrm{Id}$ and the proof follows.

Detour

Consider $\phi : \mathcal{U}_{S,K} \to \mathbb{R}^{|S|}$ given by $x \mapsto (\log|x|_v)_{v\in S}$ in category of groups. By taking log of the product formula, we see $\mathrm{im}(\phi)$ is contained in the hyperplane $\sum_{v\in S} y_v = 0$. By Kronecker's theorem, the kernel of ϕ is the group μ_K of roots of unity in K. This is part of the Dirichlet's unit theorem.

Next, recall the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$, given coordinate wise by

$$(\mathbf{x},\mathbf{y})=((x_0:\ldots:x_n),(y_0:\ldots:y_m)) \mapsto \mathbf{x}\otimes\mathbf{y} := (x_i y_j)$$

where the (ij) are ordered, e.g. lexicographically. This shows

$$h(\mathbf{x}\otimes\mathbf{y}) = h(\mathbf{x})+h(\mathbf{y})$$

using $\max_{ij}|x_i y_j|_v = \max_i |x_i|_v \cdot \max_j |y_j|_v$.

For local computations, its often convenient to introduce the following function $\log^+(x) := \max(0,\log(x))$. In particular, we see for any point $P \in \mathbb{A}^{n+1}$, which identified as $(1,P_1,\ldots,P_n) \in \mathbb{P}^n$, we have

$$h(P)=h(1:P_1:\ldots:P_n)=\sum_{v\in M_K}\max_j \log^+|x_j|_v$$

Proposition 1.2.7

Let $P^1, \ldots, P^r$ be points of $\mathbb{A}^n$, then

$$h(P^1 + \ldots + P^r) \leq h(P^1) + \ldots + h(P^r) + \log r$$

Proof. WLOG we may assume $P^i \in \mathbb{A}_K^n$ for some number field K. Then

$$h(P^1 + \ldots + P^r) = \sum_{v \in M_K} \max_j \log^+ |P_j^1 + \ldots + P_j^r|_v$$

If v is non-archimedean, then

$$|P_j^1 + \ldots + P_j^r|_v \leq \max_k |P_j^k|_v$$

If v is archimedean, by triangle inequality we see

$$|P_j^1 + \ldots + P_j^r|_v \leq |r|_v \cdot \max_k |P_j^k|_v$$

but then $\sum_{v|\infty} \log |r|_v = \log r$. Thus we see

$$h(P^1 + \ldots + P^r) \leq \log r + \sum_{v \in M_K} \max_{j,k} \log^+ |P_j^k|_v \leq \log r + \sum_k \max_j \log^+ |P_j^k|_v$$

The following result says the height is invariant under Galois action.

Proposition 1.2.8

Let $P \in \mathbb{P}^n_{\overline{\mathbb{Q}}}$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $h(P) = h(\sigma(P))$.

This result follows from the observation that σ induces a permutation on M_K.

Lemma 1.2.9

If $\alpha \in K \backslash \{0\}$ and $\lambda \in \mathbb{Q}$, then $h(\alpha^\lambda) = |\lambda| \cdot h(\alpha)$. In particular, $h(1/\alpha) = h(\alpha)$.

This result follows from the observation that $\log |\alpha|_v = \log^+ |\alpha|_v - \log^+ |1/\alpha|_v$, and now sum over all the places, we get $0 = h(\alpha) - h(1/\alpha)$.

Let $S \subseteq M_K$ be a finite set of places. For $\alpha \in K \backslash \{0\}$, we have

$$\sum_{v \in S} \log |\alpha|_v \leq h(\alpha)$$

If we use $1/\alpha$ instead of α, then the above lemma shows

$$\sum_{v \in S} \log |\alpha|_v \geq -h(\alpha)$$

This concludes the so-called fundamental inequality

$$-h(\alpha) \leq \sum_{v \in S} \log |\alpha|_v \leq h(\alpha) \tag{Eq. 1.2.1}$$

The next theorem is a very important result, namely:

Theorem 1.2.10: Northcott's Theorem

There are only finitely many algebraic numbers of bounded degree and bounded height.

Chapter 2

Weil Heights

In this chapter, we will look at heights from a geometric point of view. In particular, we will define local Weil heights associated to a Cartier divisor on projective X, and studying their properties. Then we move to the global case, and we will prove Northcott's property in this case.

2.1 Review: Cartier Divisors

2.2 Local Heights